Module V: Vector Spaces

What is a vector space?

At the end of this module, students will be able to...

V1. Vector property verification. ... show why an example satisfies a given vector space property, but does not satisfy another given property.

V2. Vector space identification. ... list the eight defining properties of a vector space, infer which of these properties a given example satisfies, and thus determine if the example is a vector space.

V3. Linear combinations. ... determine if a Euclidean vector can be written as a linear combination of a given set of Euclidean vectors.

V4. Spanning sets. ... determine if a set of Euclidean vectors spans $\mathbb{R}^n$.

V5. Subspaces. ... determine if a subset of $\mathbb{R}^n$ is a subspace or not.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Add complex numbers and multiply complex numbers by scalars.
- Add polynomials and multiply polynomials by scalars.
- Perform basic manipulations of augmented matrices and linear systems **E1,E2,E3**.

The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Acaemdy): `http://bit.ly/2y8AOwa`
- Linear combinations of Euclidean vectors (Khan Academy): `http://bit.ly/2nK3wne`
- Adding and subtracting complex numbers (Khan Academy): `http://bit.ly/1PE3ZMQ`
- Adding and subtracting polynomials (Khan Academy): `http://bit.ly/2d5SLGZ`

Module V Section 0

Activity V.0.1 *(∼20 min)*

Consider each of the following vector properties. Label each property with $\mathbb{R}^1$, $\mathbb{R}^2$, and/or $\mathbb{R}^3$ if that property holds for Euclidean vectors/scalars $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of that dimension.

1. **Addition associativity.**
   $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
2. **Addition commutivity.**
   $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. **Addition identity.**
   There exists some $\mathbf{z}$ where $\mathbf{v} + \mathbf{z} = \mathbf{v}$.
4. **Addition inverse.**
   There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}$.
5. **Addition midpoint uniqueness.**
   There exists a unique $\mathbf{m}$ where the distance from $\mathbf{u}$ to $\mathbf{m}$ equals the distance from $\mathbf{m}$ to $\mathbf{v}$.
6. **Scalar multiplication associativity.**
   $a(b\mathbf{v}) = (ab)\mathbf{v}$.
7. **Scalar multiplication identity.**
   $1\mathbf{v} = \mathbf{v}$.
8. **Scalar multiplication relativity.**
   There exists some scalar c where either $c\mathbf{v} = \mathbf{w}$ or $c\mathbf{w} = \mathbf{v}$.
9. **Scalar distribution.**
   $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
10. **Vector distribution.**
    $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.
11. **Orthogonality.**
    There exists a non-zero vector $\mathbf{n}$ such that $\mathbf{n}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
12. **Bidimensionality.**
    $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ for some value of a, b.

Definition V.0.2

A **vector space** V is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ belong to V, and let a, b be scalar numbers.

- **Addition associativity.** $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **Addition commutivity.** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- **Addition inverse.** There exists some $\mathbf{z}$ where $\mathbf{v} + \mathbf{z} = \mathbf{v}$.
- **Additive inverses exist.** There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}$.
- **Scalar multiplication associativity.** $a(b\mathbf{v}) = (ab)\mathbf{v}$.
- **Scalar multiplication identity.** $1\mathbf{v} = \mathbf{v}$.
- **Scalar distribution.** $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
- **Vector distribution.** $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Any **Euclidean vector space** $\mathbb{R}^n$ satisfies all eight requirements regardless of the value of n, but we will also study other types of vector spaces.

Module V Section 1

Remark V.1.1

Last time, we defined a **vector space** V to be any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following eight properties for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V, and all scalars (i.e. real numbers) a, b.

- **Addition associativity.** $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **Addition commutivity.** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- **Addition inverse.** There exists some $\mathbf{z}$ where $\mathbf{v} + \mathbf{z} = \mathbf{v}$.
- **Additive inverses exist.** There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}$.
- **Scalar multiplication associativity.** $a(b\mathbf{v}) = (ab)\mathbf{v}$.
- **Scalar multiplication identity.** $1\mathbf{v} = \mathbf{v}$.
- **Scalar distribution.** $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
- **Vector distribution.** $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Remark V.1.2

The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$: Euclidean vectors with n components.
- $\mathbb{R}^\infty$: Sequences of real numbers $(v_1, v_2, \ldots)$.
- $M_{m,n}$: Matrices of real numbers with m rows and n columns.
- $\mathbb{C}$: Complex numbers.
- $\mathcal{P}^n$: Polynomials of degree n or less.
- $\mathcal{P}$: Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

Activity V.1.3 *(∼20 min)*

Consider the set $V = \{(x, y) \mid y = e^x\}$ with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \qquad c \odot (x, y) = (cx, y^c)$$

Activity V.1.3 *(∼20 min)*
Consider the set $V = \{(x, y) \mid y = e^x\}$ with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \qquad c \odot (x, y) = (cx, y^c)$$

Part 1: Show that V satisfies the vector distributive property

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v})$$

by letting $\mathbf{v} = (x, y)$ and showing both sides simplify to the same expression.

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by letting $\mathbf{v} = (x, y)$ and showing both sides simplify to the same expression.
Part 2: Show that V contains an additive identity element by choosing $\mathbf{z} = (\,?\,,\,?\,)$ such that $\mathbf{v} \oplus \mathbf{z} = (x, y) \oplus (\,?\,,\,?\,) = \mathbf{v}$ for any $\mathbf{v} = (x, y) \in V$.

Remark V.1.4

It turns out $V = \{(x, y) \mid y = e^x\}$ with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \qquad c \odot (x, y) = (cx, y^c)$$

satisifes all eight properties.

- **Addition associativity.** $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$.
- **Addition commutivity.** $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$.
- **Addition identity.** There exists some $\mathbf{z}$ where $\mathbf{v} \oplus \mathbf{z} = \mathbf{v}$.
- **Addition inverse.** There exists some $-\mathbf{v}$ where $\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{z}$.
- **Scalar multiplication associativity.** $a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}$.
- **Scalar multiplication identity.** $1 \odot \mathbf{v} = \mathbf{v}$.
- **Scalar distribution.** $a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v})$.
- **Vector distribution.** $(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v})$.

Thus, V is a vector space.

Activity V.1.5 *(∼15 min)*

Let $V = \{(x, y) \,|\, x, y \in \mathbb{R}\}$ have operations defined by

$$(x, y) \oplus (z, w) = (x + y + z + w, x^2 + z^2) \qquad c \odot (x, y) = (x^c, y + c - 1).$$

Activity V.1.5 *(∼15 min)*
Let $V = \{(x, y) \,|\, x, y \in \mathbb{R}\}$ have operations defined by

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Part 1: Show that the scalar multiplication identity holds by simplifying $1 \odot (x, y)$ to (x, y).

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Part 1: Show that the scalar multiplication identity holds by simplifying $1 \odot (x, y)$ to (x, y).

Part 2: Show that the addition identity property fails by showing that $(0, -1) \oplus \mathbf{z} \neq (0, -1)$ no matter how $\mathbf{z} = (z_1, z_2)$ is chosen.

Activity V.1.5 *(~15 min)*

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Part 1: Show that the scalar multiplication identity holds by simplifying $1 \odot (x, y)$ to (x, y).

Part 2: Show that the addition identity property fails by showing that $(0, -1) \oplus \mathbf{z} \neq (0, -1)$ no matter how $\mathbf{z} = (z_1, z_2)$ is chosen.

Part 3: Can V be a vector space?

Definition V.1.6

A **linear combination** of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is given by $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$ for any choice of scalar multiples $c_1, c_2, \dots, c_m$.

For example, we can say $\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ since

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Definition V.1.7

The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m \,|\, c_i \in \mathbb{R}\}.$$

For example:

$$\operatorname{span}\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\-1\\2\end{bmatrix} + b\begin{bmatrix}1\\2\\1\end{bmatrix} \,\middle|\, a, b \in \mathbb{R}\right\}$$

Activity V.1.8 *(∼10 min)*

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

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Part 1: Sketch $1\begin{bmatrix}1\\2\end{bmatrix}$, $3\begin{bmatrix}1\\2\end{bmatrix}$, $0\begin{bmatrix}1\\2\end{bmatrix}$, and $-2\begin{bmatrix}1\\2\end{bmatrix}$ in the xy plane.

Activity V.1.8 *(∼10 min)*

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

Part 1: Sketch $1\begin{bmatrix}1\\2\end{bmatrix}$, $3\begin{bmatrix}1\\2\end{bmatrix}$, $0\begin{bmatrix}1\\2\end{bmatrix}$, and $-2\begin{bmatrix}1\\2\end{bmatrix}$ in the xy plane.

Part 2: Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}=\left\{a\begin{bmatrix}1\\2\end{bmatrix}\,\middle|\, a\in\mathbb{R}\right\}$ in the xy plane.

Activity V.1.9 *(∼10 min)*

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Activity V.1.9 *(∼10 min)*

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Part 1: Sketch the following linear combinations in the xy plane.

$$1\begin{bmatrix}1\\2\end{bmatrix}+0\begin{bmatrix}-1\\1\end{bmatrix} \qquad 0\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix} \qquad 1\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}$$

$$-2\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix} \qquad -1\begin{bmatrix}1\\2\end{bmatrix}+-2\begin{bmatrix}-1\\1\end{bmatrix}$$

Activity V.1.9 *(∼10 min)*

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Part 1: Sketch the following linear combinations in the xy plane.

$$1\begin{bmatrix}1\\2\end{bmatrix}+0\begin{bmatrix}-1\\1\end{bmatrix}\qquad 0\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}\qquad 1\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}$$

$$-2\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}\qquad -1\begin{bmatrix}1\\2\end{bmatrix}+-2\begin{bmatrix}-1\\1\end{bmatrix}$$

Part 2: Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$ in the xy plane.

Activity V.1.10 *(∼5 min)*

Sketch a representation of all the vectors belonging to span $\left\{\begin{bmatrix}6\\-4\end{bmatrix},\begin{bmatrix}-3\\2\end{bmatrix}\right\}$ in the xy plane.

Module V Section 2

Remark V.2.1

Recall these definitions from last class:

- A **linear combination** of vectors is given by adding scalar multiples of those vectors, such as:

$$\begin{bmatrix}3\\0\\5\end{bmatrix} = 2\begin{bmatrix}1\\-1\\2\end{bmatrix} + 1\begin{bmatrix}1\\2\\1\end{bmatrix}$$

- The **span** of a set of vectors is the collection of all linear combinations of that set, such as:

$$\operatorname{span}\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\-1\\2\end{bmatrix} + b\begin{bmatrix}1\\2\\1\end{bmatrix} \,\middle|\, a, b \in \mathbb{R}\right\}$$

Activity V.2.2 *(∼15 min)*

The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when there exists a solution to the vector equation $x_1\begin{bmatrix}1\\0\\-3\end{bmatrix}+x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix}=\begin{bmatrix}-1\\-6\\1\end{bmatrix}$.

Activity V.2.2 *(~15 min)*

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when there exists a solution to the vector equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$.

Part 1: Reinterpret this vector equation as a system of linear equations.

Activity V.2.2 *(∼15 min)*

The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when there exists a solution to the vector equation $x_1\begin{bmatrix}1\\0\\-3\end{bmatrix}+x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix}=\begin{bmatrix}-1\\-6\\1\end{bmatrix}$.

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Find its solution set, using CoCalc.com to find RREF of its corresponding augmented matrix.

Activity V.2.2 *(∼15 min)*

The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when there exists a solution to the vector equation $x_1\begin{bmatrix}1\\0\\-3\end{bmatrix} + x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix} = \begin{bmatrix}-1\\-6\\1\end{bmatrix}$.

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Find its solution set, using CoCalc.com to find RREF of its corresponding augmented matrix.

Part 3: Given this solution set, does $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belong to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$?

Fact V.2.3

A vector $\mathbf{b}$ belongs to span$\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ if and only if the linear system corresponding to $[\mathbf{v}_1 \ \ldots \ \mathbf{v}_n \,|\, \mathbf{b}]$ is consistent.

Put another way, $\mathbf{b}$ belongs to span$\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ exactly when RREF$[\mathbf{v}_1 \ \ldots \ \mathbf{v}_n \,|\, \mathbf{b}]$ doesn't have a row $[0 \cdots 0 \,|\, 1]$ representing the contradiction $0 = 1$.

Activity V.2.4 *(~10 min)*

Determine if $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 5 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

Activity V.2.5 *(∼5 min)*

Determine if $\begin{bmatrix}-1\\-9\\0\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ by row-reducing an appropriate matrix.

Activity V.2.6 *(∼10 min)*
Does the third-degree polynomial $3y^3 - 2y^2 + y + 5$ in $\mathcal{P}^3$ belong to $\operatorname{span}\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$?

Activity V.2.6 *($\sim$10 min)*
Does the third-degree polynomial $3y^3 - 2y^2 + y + 5$ in $\mathcal{P}^3$ belong to $\text{span}\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$?
Part 1: Reinterpret this question as an equivalent exercise involving Euclidean vectors in $\mathbb{R}^4$. (Hint: What four numbers must you know to write a $\mathcal{P}^3$ polynomial?)

Activity V.2.6 *(∼10 min)*

Does the third-degree polynomial $3y^3 - 2y^2 + y + 5$ in $\mathcal{P}^3$ belong to $\operatorname{span}\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$?

Part 1: Reinterpret this question as an equivalent exercise involving Euclidean vectors in $\mathbb{R}^4$. (Hint: What four numbers must you know to write a $\mathcal{P}^3$ polynomial?)

Part 2: Solve this equivalent exercise, and use its solution to answer the original question.

Activity V.2.7 *(∼5 min)*

Does the matrix $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$ belong to span $\left\{ \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix} \right\}$?

Activity V.2.8 *(∼5 min)*

Does the complex number $2i$ belong to $\operatorname{span}\{-3+i, 6-2i\}$?

Module V Section 3

Activity V.3.1 *(∼5 min)*

How many vectors are required to span $\mathbb{R}^2$? Sketch a drawing in the xy plane to support your answer.

(a) 1

(b) 2

(c) 3

(d) 4

(e) Infinitely Many

Activity V.3.2 *(∼5 min)*

How many vectors are required to span $\mathbb{R}^3$?

(a) 1

(b) 2

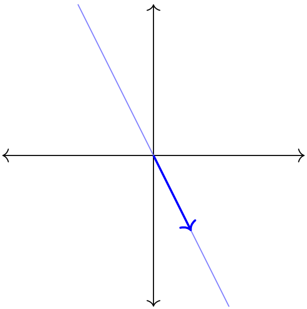
(c) 3

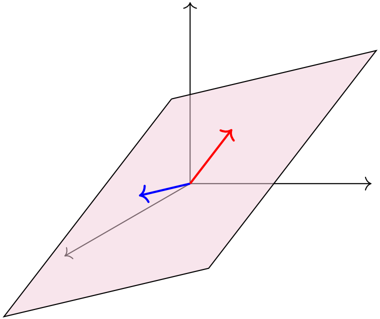
(d) 4

(e) Infinitely Many

Fact V.3.3

At least n vectors are required to span $\mathbb{R}^n$.





Activity V.3.4 *(∼15 min)*

Choose a vector $\begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in $\mathbb{R}^3$ that is not in span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ by using CoCalc to verify that RREF $\left[\begin{array}{cc|c} 1 & -2 & ? \\ -1 & 0 & ? \\ 0 & 1 & ? \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$. (Why does this work?)

Fact V.3.5

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ fails to span all of $\mathbb{R}^n$ exactly when $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_m]$ has a row of zeros:

$$\begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & a \\ -1 & 0 & b \\ 0 & 1 & c \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \text{ for some choice of vector } \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Activity V.3.6 *(∼5 min)*

Consider the set of vectors $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$. Does $\mathbb{R}^4 = \operatorname{span} S$?

Activity V.3.7 *(∼10 min)*

Consider the set of third-degree polynomials

$$S = \{2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16, -x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2$$

Does $\mathcal{P}^3 = \operatorname{span} S$? (Hint: first rewrite the question so it is about Euclidean vectors.)

Activity V.3.8 *(∼10 min)*
Consider the set of matrices

$$S = \left\{ \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

Does $M_{2,2} = \operatorname{span} S$?

Activity V.3.9 *(∼10 min)*

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^7$ be three vectors, and suppose $\mathbf{w}$ is another vector with $\mathbf{w} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. What can you conclude about $\operatorname{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

(a) $\operatorname{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is larger than $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) $\operatorname{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

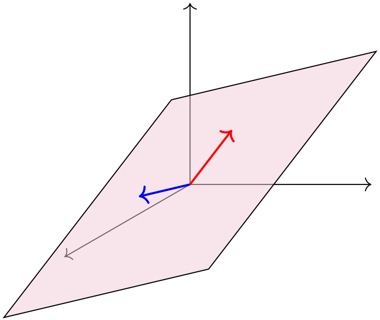
(c) $\operatorname{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is smaller than $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Module V Section 4

Definition V.4.1

A subset of a vector space is called a **subspace** if it is a vector space on its own.

For example, the span of these two vectors forms a planar subspace inside of the larger vector space $\mathbb{R}^3$.



Fact V.4.2

Any sub**set** S of a vector space V satisfies the eight vector space properties automatically, since it is a collection of known vectors.

However, to verify that it's a sub**space**, we need to check that addition and multiplication still make sense using only vectors from S. So we need to check two things:

- The set is **closed under addition**: for any $\mathbf{x}, \mathbf{y} \in S$, the sum $\mathbf{x} + \mathbf{y}$ is also in S.
- The set is **closed under scalar multiplication**: for any $\mathbf{x} \in S$ and scalar $c \in \mathbb{R}$, the product $c\mathbf{x}$ is also in S.

Activity V.4.3 *(∼15 min)*

Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x + 2y + z = 0 \right\}$.

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Part 1: Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S, so $x + 2y + z = 0$ and $a + 2b + c = 0$. Show that $\mathbf{v} + \mathbf{w} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$ also belongs to S by verifying that $(x + a) + 2(y + b) + (z + c) = 0$.

Activity V.4.3 *(∼15 min)*

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Part 2: Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$, so $x + 2y + z = 0$. Show that $c\mathbf{v}$ also belongs to S for any $c \in \mathbb{R}$.

Activity V.4.3 *(∼15 min)*

Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + 2y + z = 0 \right\}$.

Part 1: Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S, so $x + 2y + z = 0$ and $a + 2b + c = 0$. Show that $\mathbf{v} + \mathbf{w} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$ also belongs to S by verifying that $(x + a) + 2(y + b) + (z + c) = 0$.

Part 2: Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$, so $x + 2y + z = 0$. Show that $c\mathbf{v}$ also belongs to S for any $c \in \mathbb{R}$.

Part 3: Is S is a subspace of $\mathbb{R}^3$?

Activity V.4.4 *(~10 min)*

Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x + 2y + z = 4 \right\}$. Choose a vector $\mathbf{v} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in S and a real number $c = ?$, and show that $c\mathbf{v}$ isn't in S. Is S a subspace of $\mathbb{R}^3$?

Remark V.4.5

Since 0 is a scalar and $0\mathbf{v} = \mathbf{z}$ for any vector $\mathbf{v}$, a set that is closed under scalar multiplication must contain the zero vector $\mathbf{z}$ for that vector space.

Put another way, an easy way to check that a subset isn't a subspace is to show it doesn't contain $\mathbf{0}$.

Activity V.4.6 *(∼10 min)*

Consider these two subsets of $\mathbb{R}^4$:

$$S = \left\{ \begin{bmatrix} a \\ b \\ -b \\ -a \end{bmatrix} \,\middle|\, a, b \text{ are real numbers} \right\} \qquad T = \left\{ \begin{bmatrix} a \\ b \\ b-1 \\ a-1 \end{bmatrix} \,\middle|\, a, b \text{ are real numbers} \right\}$$

Activity V.4.6 *(∼10 min)*

Consider these two subsets of $\mathbb{R}^4$:

$$S = \left\{ \begin{bmatrix} a \\ b \\ -b \\ -a \end{bmatrix} \,\middle|\, a, b \text{ are real numbers} \right\} \qquad T = \left\{ \begin{bmatrix} a \\ b \\ b-1 \\ a-1 \end{bmatrix} \,\middle|\, a, b \text{ are real numbers} \right\}$$

Part 1: Which set is not a subspace of $\mathbb{R}^4$?

Activity V.4.6 *(∼10 min)*

Consider these two subsets of $\mathbb{R}^4$:

$$S = \left\{ \begin{bmatrix} a \\ b \\ -b \\ -a \end{bmatrix} \middle| \, a, b \text{ are real numbers} \right\} \qquad T = \left\{ \begin{bmatrix} a \\ b \\ b-1 \\ a-1 \end{bmatrix} \middle| \, a, b \text{ are real numbers} \right\}$$

Part 1: Which set is not a subspace of $\mathbb{R}^4$?

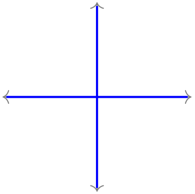
Part 2: Is the set of polynomials

$$S = \left\{ ax^3 + bx^2 + (b-1)x + (a-1) \,\middle|\, a, b \text{ are real numbers} \right\}$$

a subspace of $\mathcal{P}^3$?

Activity V.4.7 *(∼10 min)*

Consider the subset A of $\mathbb{R}^2$ where at least one coordinate of each vector is 0.



This set contains $\mathbf{0}$, and it's not hard to show that for every $\mathbf{v}$ in A and scalar $c \in \mathbb{R}$, $c\mathbf{v}$ is also in A. Is A a subspace of $\mathbb{R}^2$? Why?

Activity V.4.8 *(∼5 min)*

Let W be a subspace of a vector space V. How are span W and W related?

(a) span W is bigger than W

(b) span W is the same as W

(c) span W is smaller than W

Fact V.4.9

If S is any subset of a vector space V, then since $\operatorname{span} S$ collects all possible linear combinations, $\operatorname{span} S$ is automatically a subspace of V.

In fact, $\operatorname{span} S$ is always the smallest subspace of V that contains all the vectors in S.