① Evaluate $\sinh(\ln 6)$

$$= \frac{e^{\ln 6} - e^{-\ln 6}}{2} = \frac{6 - \frac{1}{6}}{2}$$

$$= \frac{36/6 - 1/6}{2} = \frac{35/6}{2} = \boxed{35/12}$$

② Prove that $\cosh(2x) = \cosh^2 x + \sinh^2 x$.

$$\cosh(2x) = \frac{e^{2x} + e^{-2x}}{2}$$

$$\cosh^2 x + \sinh^2 x = \left(\frac{e^{x} + e^{-x}}{2}\right)^2 + \left(\frac{e^{x} - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2\cancel{e^{x}e^{-x}} + e^{-2x}}{4} + \frac{e^{2x} - 2\cancel{e^{x}e^{-x}} + e^{-2x}}{4}$$

$$= \frac{2e^{2x} + 2e^{-2x}}{4}$$

$$= \frac{e^{2x} + e^{-2x}}{4} \qquad \square$$

③ Prove that $\cosh^2 x - \sinh^2 x = 1$.

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2\cancel{e^x e^{-x}} + e^{-2x}}{4} - \frac{e^{2x} - 2\cancel{e^x e^{-x}} + e^{-2x}}{4}$$

$$= \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}} - \cancel{e^{2x}} + 2 - \cancel{e^{-2x}}}{4}$$

$$= \frac{4}{4} = 1. \quad \square$$

OR

(Using derivatives!)

$$\frac{d}{dx}\left[\cosh^2 x - \sinh^2 x\right] = 2\cancel{\cosh x}(\cancel{\sinh x}) - 2\cancel{\sinh x}(\cancel{\cosh x})$$
$$= 0$$

Thus $\cosh^2 x - \sinh^2 x = C$.

When $x = 0$,

$$\cosh^2(0) - \sinh^2(0) = C$$
$$1 - 0 = C$$
$$C = 1.$$

Therefore $\cosh^2 x - \sinh^2 x = 1$. $\square$

(4) Evaluate $\tanh(\ln 3)$.

$$= \frac{e^{\ln 3} - e^{-\ln 3}}{e^{\ln 3} + e^{-\ln 3}} = \frac{3 - \frac{1}{3}}{3 + \frac{1}{3}} = \frac{8/\cancel{3}}{10/\cancel{3}} = \frac{8}{10} = \boxed{\frac{4}{5}}$$

(5) Simplify $\sinh(x)\coth(x)\cosh(x) - \frac{1}{\operatorname{csch}^2(x)}$.

$$= \cancel{\sinh(x)}\frac{\cosh(x)}{\cancel{\sinh(x)}}\cosh(x) - \frac{1}{\frac{1}{\sinh^2(x)}}$$

$$= \cosh^2(x) - \sinh^2(x)$$

By #3,

$$= \boxed{1}$$

(6) Prove that $\frac{d}{dx}\left[\sinh x\right] = \cosh x$.

$$\frac{d}{dx}\left[\sinh x\right] = \frac{d}{dx}\left[\frac{e^{x}-e^{-x}}{2}\right]$$
$$= \frac{1}{2}\frac{d}{dx}\left[e^{x}-e^{-x}\right]$$
$$= \frac{1}{2}\left(e^{x}-e^{-x}(-1)\right)$$
$$= \frac{e^{x}+e^{-x}}{2} = \cosh(x). \quad \square$$

(7) Prove that $\frac{d}{dx}\left[\operatorname{sech} x\right] = -\operatorname{sech} x \tanh x$.

$$\frac{d}{dx}\left[\operatorname{sech} x\right] = \frac{d}{dx}\left[\frac{1}{\cosh x}\right] = \frac{\cancel{(\cosh x)(0)} - (1)(\sinh x)}{(\cosh x)^2}$$

$$= -\frac{1}{\cosh x}\frac{\sinh x}{\cosh x}$$

$$= -\operatorname{sech} x \tanh x. \quad \square$$

(8) Compute $\frac{d}{dx}\left[\tanh(3x) - \operatorname{sech}(\ln x)\right]$.

$$= \operatorname{sech}^2(3x)(3) - \left(-\operatorname{sech}(\ln x)\tanh(\ln x)\left(\frac{1}{x}\right)\right)$$

$$= \boxed{3\operatorname{sech}^2(3x) + \frac{\operatorname{sech}(\ln x)\tanh(\ln x)}{x}}$$

(9) Find $\int 3\operatorname{csch} x \coth x - 2\sinh x \, dx$.

$$= 3(-\operatorname{csch} x) - 2(\cosh x) + C$$

$$= \boxed{-3\operatorname{csch} x - 2\cosh x + C}$$

(10) Let $\sinh^{\leftarrow}(x)$ be the inverse of $\sinh(x)$.

Use $\frac{d}{dx}\left[f^{\leftarrow}(x)\right] = \frac{1}{f'(f^{\leftarrow}(x))}$ to prove

$$\frac{d}{dx}\left[\sinh^{\leftarrow}(x)\right] = \frac{1}{\sqrt{1+x^2}}.$$

$$\frac{d}{dx}\left[\sinh^{\leftarrow}(x)\right] = \frac{1}{\cosh(\sinh^{\leftarrow}(x))}$$

(Recall $\cosh^2 y - \sinh^2 y = 1$ implies $\cosh^2 y = 1+\sinh^2 y$,

and since $\cosh^2 y = \frac{e^y+e^{-y}}{2} > 0$, $\cosh y = +\sqrt{1+\sinh^2 y}$.)

$$= \frac{1}{\sqrt{1+\sinh^2(\cancel{\sinh^{\leftarrow}} x)}}$$

$$= \frac{1}{\sqrt{1+x^2}}. \qquad \square$$

(11) Prove $\sinh^{-1}(x) = \ln\left(\sqrt{x^2+1}+x\right)$.

$$\frac{d}{dx}\left[\sinh^{-1}(x)\right] = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}\left[\ln\left(\sqrt{x^2+1}+x\right)\right] = \frac{1}{\sqrt{x^2+1}+x}\left(\cancel{\tfrac{1}{2}}(x^2+1)^{-1/2}(\cancel{2}x)+1\right)$$

$$= \frac{1}{\sqrt{x^2+1}+x}\left(\frac{x}{\sqrt{x^2+1}}+\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{\cancel{\sqrt{x^2+1}+x}}\left(\frac{\cancel{x+\sqrt{x^2+1}}}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{\sqrt{x^2+1}}$$

Thus $\sinh^{-1}(x) = \ln\left(\sqrt{x^2+1}+x\right) + C$

Let $x=0$, then

$$\cancel{\sinh^{-1}(0)} = \cancel{\ln(1+0)} + C$$

$$0 = C$$

Therefore $\sinh^{-1}(x) = \ln\left(\sqrt{x^2+1}+x\right)$. $\square$