① Find $\int \frac{z}{\sqrt{1+4z^2}}\,dz$

Let $1+4z^2 = 1+\tan^2\theta = \sec^2\theta \longrightarrow \sec\theta = \sqrt{1+4z^2}$

$4z^2 = \tan^2\theta$

$2z = \tan\theta$

$z = \frac{1}{2}\tan\theta$

$dz = \frac{1}{2}\sec^2\theta\, d\theta$

$= \int \frac{\cancel{z}}{\sqrt{\cancel{\sec^2\theta}}} \cancel{\frac{1}{z}} \sec^2\theta\, d\theta$

$= \int \sec\theta\, d\theta$

$= \ln|\sec\theta + \tan\theta| + C$

$= \boxed{\ln\left|\sqrt{1+4z^2} + 2z\right| + C}$

(2) Find $\int \frac{x^3}{9+x^2}\,dx$.

Let

$9+x^2 = 9+9\tan^2\theta = 9\sec^2\theta \longrightarrow \sec\theta = \sqrt{1+\frac{1}{9}x^2}$

$x^2 = 9\tan^2\theta$

$x = 3\tan\theta \longrightarrow \tan\theta = \frac{x}{3}$

$dx = 3\sec^2\theta\, d\theta$

$$= \int \frac{\overset{3}{\cancel{27}}\tan^3\theta}{\cancel{9}\cancel{\sec^2\theta}}\, 3\cancel{\sec^2\theta}\, d\theta$$

$$= 9\int \tan\theta(\sec^2\theta - 1)\, d\theta$$

$$= 9\int \tan\theta\sec^2\theta\, d\theta - 9\int \tan\theta\, d\theta$$

$$= \frac{9}{2}\tan^2\theta - 9\ln|\sec\theta| + C$$

$$= \frac{9}{2}\left(\frac{x}{3}\right)^2 - 9\ln\sqrt{1+\frac{1}{9}x^2} + C$$

$$= \boxed{\frac{1}{2}x^2 - \frac{9}{2}\ln\left(1+\frac{1}{9}x^2\right) + C}$$

③ Find $\int \frac{4}{(1-y^2)^{3/2}}\,dy$.

Let $1-y^2 = 1-\sin^2\theta = \cos^2\theta \rightarrow \cos\theta = \sqrt{1-y^2}$

$y^2 = \sin^2\theta$

$y = \sin\theta$

$dy = \cos\theta\, d\theta$

$$= \int \frac{4}{(\cos^2\theta)^{3/2}} \cos\theta\, d\theta$$

$$= \int \frac{4}{\sqrt{\cos\theta}^{\cancel{3/2}}} \cancel{\cos\theta}\, d\theta$$

$$= \int 4\sec^2\theta\, d\theta$$

$$= 4\tan\theta + C$$

$$= \boxed{\frac{4y}{\sqrt{1-y^2}} + C}$$

4. Find $\int \frac{2x^3}{\sqrt{9-x^2}}\,dx$.

Let $9-x^2 = 9-9\sin^2\theta = 9\cos^2\theta \searrow \cos\theta = \sqrt{1-\frac{1}{9}x^2}$

$x^2 = 9\sin^2\theta$

$x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$$= \int \frac{2(3\sin\theta)^3}{\cancel{\sqrt{9\cos^2\theta}}}\,\cancel{3\cos\theta}\,d\theta$$

$$= 54\int \sin^3\theta\,d\theta$$

$$= 54\int \sin\theta\,d\theta - 54\int \cos^2\theta\sin\theta\,d\theta$$

$$= -54\cos\theta + 18\cos^3\theta + C$$

$$= \boxed{-54\sqrt{1-\tfrac{1}{9}x^2} + 18\left(1-\tfrac{1}{9}x^2\right)^{3/2} + C}$$

5 Prove $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{\leftarrow} x + C$.

Let $1 - x^2 = 1 - \sin^2\theta = \cos^2\theta$

$x^2 = \sin^2\theta$

$x = \sin\theta \longrightarrow \theta = \sin^{\leftarrow} x$

$dx = \cos\theta\, d\theta$

$$= \int \frac{1}{\cancel{\sqrt{\cos^2\theta}}} \cancel{\cos\theta}\, d\theta$$

$$= \int 1\, d\theta$$

$$= \theta + C$$

$$= \boxed{\sin^{\leftarrow} x + C}$$