7. Does $\sum_{n=-2}^{\infty} \frac{1}{e^n}$ converge or diverge?

$$\int_0^\infty \frac{1}{e^x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx$$

Let $u = -x$, $du = -dx$; $x = b \Rightarrow u = -b$, $x = 0 \Rightarrow u = 0$

$$= \lim_{b\to\infty} \int_0^{-b} -e^u\,du$$

$$= \lim_{b\to\infty} \left[-e^u\right]_0^{-b}$$

$$= \left(\lim_{b\to\infty} -\frac{1}{e^b}\right) + e^0$$

$$= 0 + 1 = 1$$

Since the integral converges, the series also $\boxed{\text{converges}}$.

OR Geo. Series

$$= \sum_{n=-2}^{\infty} \left(\frac{1}{e}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^{n-2}$$

$$= \sum_{n=0}^{\infty} (e^2)\left(\frac{1}{e}\right)^n$$

$$= \boxed{\frac{e^2}{1-1/e}} \quad \boxed{\text{Converges}}$$

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8. Show that $\int_1^\infty \frac{1}{x^2}\,dx \neq \sum_{n=1}^{\infty} \frac{1}{n^2}$, even though they both converge.

From 4, $\int_1^\infty \frac{1}{x^2}\,dx = 1$. Thus $\sum_{n=1}^{\infty} \frac{1}{n^2}$ also converges.

Thus
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \left(\frac{1}{4} + \frac{1}{9} + \dots\right)$$
$$> \frac{1}{1} = 1 = \int_1^\infty \frac{1}{x^2}\,dx. \qquad \square$$

(Fun Fact

In advanced calculus, it can be shown that
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.64493.)$$

(9) Does $\sum_{k=100}^{\infty} \frac{5}{\sqrt[7]{k^6}}$ converge or diverge?

$$= 5 \sum_{k=100}^{\infty} \frac{1}{k^{6/7}} \quad p$$

Since $p = 6/7 \leq 1$, the series $\boxed{\text{diverges}}$ by the p-Series Test.

(10) Does $\sum_{n=5}^{\infty} \frac{1}{n^2-8n+16}$ converge or diverge?

$$= \sum_{n=5}^{\infty} \frac{1}{(n-4)^2}$$

$$= \sum_{n=5-4}^{\infty} \frac{1}{((n+\cancel{4})\cancel{-4})^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \quad p$$

Since $p = 2 > 1$, the series $\boxed{\text{converges}}$ by the p-Series Test.

(11) Does $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$ converge or diverge?

$$\int_0^\infty \frac{e^x}{1+(e^x)^2}\,dx = \lim_{b\to\infty} \int_0^b \frac{e^x}{1+(e^x)^2}\,dx$$

Let $u = e^x$ $\quad x = b \Rightarrow u = e^b$

$du = e^x dx$ $\quad x = 0 \Rightarrow u = 1$

$$= \lim_{b\to\infty} \int_1^{e^b} \frac{1}{1+u^2}\,du$$

$$= \lim_{b\to\infty} \left[\tan^{-1}(u)\right]_1^{e^b}$$

$$= \left(\lim_{b\to\infty} \tan^{-1}(e^b)\right) - \tan^{-1}(1)$$

$$= \pi/2 - \pi/4$$

$$= \pi/4$$

Since the integral converges, the series also

$\boxed{\text{converges}}$.

(12) Does $\sum_{n=0}^{\infty} \frac{2n}{(n^2+1)^2}$ converge or diverge?

$$\int_0^{\infty} \frac{2x}{(x^2+1)^2}dx = \lim_{b\to\infty} \int_0^{b} \frac{2x}{(x^2+1)^2}dx$$

Let $u = x^2+1$, $du = 2x\,dx$

$x = b \Rightarrow u = b^2+1$

$x = 0 \Rightarrow u = 0^2+1 = 1$

$$= \lim_{b\to\infty} \int_1^{b^2+1} \frac{1}{u^2}du$$

$$= \lim_{b\to\infty} \left[-\frac{1}{u}\right]_1^{b^2+1}$$

$$= \lim_{b\to\infty} \left[-\frac{1}{b^2+1}\right] + \frac{1}{1}$$

$$= 0 + 1 = 1$$ converges

Since $\int_0^{\infty} \frac{2x}{(x^2+1)^2}dx$ converges, the series $\sum_{n=0}^{\infty} \frac{2n}{(n^2+1)^2}$ also $\boxed{\text{converges}}$.

(13) Does $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$ converge or diverge?

$$= \sum_{n=2-1}^{\infty} \frac{1}{\sqrt{n+\cancel{1-1}}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\underset{p}{1/2}}}$$

$\boxed{\text{diverges}}$ as a p-Series since $p \geq 1$.

OR

$$\int_2^{\infty} \frac{1}{\sqrt{x-1}}\,dx = \lim_{b\to\infty} \int_2^b (x-1)^{-1/2}\,dx$$

$$= \lim_{b\to\infty} \left[2(x-1)^{1/2}\right]_2^b$$

$$= \left(\lim_{b\to\infty} 2\sqrt{b-1}\right) - 2\sqrt{2-1} \quad \text{diverges}$$

Since the integral diverges, the series $\boxed{\text{diverges}}$ also.