① Does $\sum_{n=0}^{\infty} \sqrt{\frac{n}{n^4+7}}$ converge or diverge?

DCT (Similar to

$$\sum \sqrt{\frac{n}{n^4}} = \sum \sqrt{\frac{1}{n^3}} = \sum \frac{1}{n^{3/2}} \leftarrow \text{Convergent p-series}$$)

$$\sqrt{\frac{n}{n^4+7}} \underset{\text{smaller or same}}{\overset{\text{bigger or same}}{\leq}} \sqrt{\frac{n}{n^4}} = \sqrt{\frac{1}{n^3}} = \frac{1}{n^{3/2}}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, the smaller $\sum_{n=0}^{\infty} \sqrt{\frac{n}{n^4+7}}$ also $\boxed{\text{converges.}}$

LCT Compare with $\sum \frac{1}{n^{3/2}}$ (Convergent p-series)

$$\lim_{n\to\infty} \frac{\sqrt{\frac{n}{n^4+7}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \sqrt{n^3}\sqrt{\frac{n}{n^4+7}}$$

$$= \sqrt{\lim_{n\to\infty} \frac{n^4}{n^4+7}}$$

$$= \sqrt{1} = 1 \leftarrow \text{between } 0 \text{ \& } \infty$$

Therefore both series $\boxed{\text{Converge}}$.

② Does $\sum_{n=3}^{\infty} \frac{4}{n^{0.8}-1}$ converge or diverge?

$\boxed{\text{DCT}}$ (Similar to $\sum \frac{4}{n^{0.8}} \leftarrow$ divergent p-series)

smaller or same

$$\frac{4}{n^{0.8}-1} \geq \frac{4}{n^{0.8}}$$

bigger or same

Since $\sum_{n=1}^{\infty} \frac{4}{n^{0.8}}$ diverges, the bigger $\sum_{n=3}^{\infty} \frac{4}{n^{0.8}-1}$ also $\boxed{\text{diverges}}$.

$\boxed{\text{LCT}}$ Compare with $\sum \frac{1}{n^{0.8}} \leftarrow$ divergent p-series

$$\lim_{n\to\infty} \frac{\frac{4}{n^{0.8}-1}}{\frac{1}{n^{0.8}}} = \lim_{n\to\infty} \frac{4n^{0.8}}{n^{0.8}-1} = \lim_{n\to\infty} \frac{\cancel{n^{0.8}}}{\cancel{n^{0.8}}} \frac{4}{1-\frac{1}{n^{0.8}}} = 4$$

$\uparrow$ between 0 & ∞

Thus both series $\boxed{\text{diverge}}$.

(3) Does $\sum_{j=2}^{\infty} \frac{e^j}{e^{2j}+1}$ converge or diverge?

$\boxed{DCT}$ Similar to $\sum \frac{e^j}{e^{2j}} = \sum \left(\frac{1}{e}\right)^j$ ← Conv. geo series

bigger or same

$$\frac{e^j}{e^{2j}+1} \leq \frac{e^j}{e^{2j}} = \frac{1}{e^j}$$

smaller or same

Since $\sum_{j=0}^{\infty} \left(\frac{1}{e}\right)^j$ converges, the smaller $\sum_{j=2}^{\infty} \frac{e^j}{e^{2j}+1}$

also $\boxed{\text{converges}}$.

$\boxed{LCT}$ Compare with $\sum_{j=0}^{\infty} \frac{e^j}{e^{2j}} = \sum_{j=0}^{\infty} \frac{1}{e^j}$ ← conv geo. series

$$\lim_{j\to\infty} \frac{\frac{e^j}{e^{2j}+1}}{\frac{1}{e^j}} = \lim_{j\to\infty} \frac{e^{2j}}{e^{2j}+1} = \lim_{j\to\infty} \frac{\cancel{e^{2j}}}{\cancel{e^{2j}}} \frac{1}{1+\cancel{1/e^{2j}}} = 1$$

← between 0 & ∞

Thus both $\boxed{\text{converge}}$

④ Does $\sum_{k=10}^{\infty} \frac{\sin^2(k)}{k^3}$ converge or diverge?

DCT (Similar to $\sum \frac{1}{k^{3}}$ since $0 \le \sin^2(k) \le 1$

convergent p-series)

bigger or same

$$\frac{\sin^2(k)}{k^3} \le \frac{1}{k^3}$$

smaller or same

Since $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges, the smaller $\sum_{k=10}^{\infty} \frac{\sin^2(k)}{k^3}$

also $\boxed{\text{converges}}$.

(5) Does $\sum_{n=4}^{\infty} \frac{1}{\ln n}$ converge or diverge?

DCT (Perhaps it's similar to $\sum \frac{1}{n}$?

divergent Harmonic series)

smaller or same

$$\frac{1}{\ln n} \geq \frac{1}{n}$$

bigger or same

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the bigger $\sum_{n=1}^{\infty} \frac{1}{\ln n}$ also

$\boxed{\text{diverges}}$.

⑥ Does $\sum_{n=4}^{\infty} \frac{5}{2n+3}$ converse or diverge?

DCT (Similar to $\sum \frac{1}{n}$ ← divergent)

smaller or same

$$\frac{5}{2n+3} \geq \frac{5}{2n+n} = \frac{5/3}{n}$$

bigger or same

Since $5/3 \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the larger $\sum_{n=4}^{\infty} \frac{5}{2n+3}$ also $\boxed{\text{diverges}}$.

LCT Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$ (divergent).

$$\lim_{n\to\infty} \frac{\frac{5}{2n+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{5n}{2n+3} = \frac{5}{2}$$ ← between 0 & ∞

Thus both $\boxed{\text{diverge}}$.

7. Does $\sum_{n=1}^{\infty} \frac{1}{1+2+\cdots+n}$ converge or diverge?

$$\frac{1}{1+2+\cdots+n} = \frac{2}{\underbrace{\begin{matrix}1+2+\cdots+n \\ +n+(n-1)+\cdots+1\end{matrix}}_{n\text{-many}}} = \frac{2}{n(n+1)}$$

$\boxed{\text{LCT}}$ Compare with $\sum \frac{1}{n^2}$ (convergent p-series)

$$\lim_{n\to\infty} \frac{\frac{2}{n(n+1)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2n^2}{n^2+n}$$

$$= \lim_{n\to\infty} \frac{2}{1+\cancel{1/n}}$$

$$= 2 \leftarrow \text{between } 0 \text{ \& } \infty$$

Both series $\boxed{\text{converge}}$.

⑧ Does $\sum_{n=0}^{\infty} \frac{2n}{(n^2+1)^2}$ converge or diverge?

DCT (Similar to $\sum \frac{2n}{(n^2)^2} = \sum \frac{2}{n^3} \leftarrow$ convergent p-series)

$$\frac{2n}{(n^2+1)^2} \underset{\text{smaller or same}}{\overset{\text{bigger or same}}{\leq}} \frac{2n}{(n^2)^2} = \frac{2n}{n^4} = \frac{2}{n^3}$$

Since $\sum_{n=1}^{\infty} \frac{2}{n^3}$ converges, the smaller $\sum_{n=0}^{\infty} \frac{2n}{(n^2+1)^2}$

also $\boxed{\text{converges}}$.

LCT Compare with convergent $\sum \frac{1}{n^3}$.

$$\lim_{n\to\infty} \frac{\frac{2n}{(n^2+1)^2}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{2n^4}{n^4+2n^2+1} = 2 \leftarrow \text{between } 0 \text{ \& } \infty$$

Thus both $\boxed{\text{converge}}$.

(9) Does $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2+3}}$ converge or diverge?

$\boxed{\text{DCT}}$ (Similar to $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n^2}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ ← divergent p-series)

smaller or same

$$\sqrt{\frac{n+1}{n^2+3}} \geq \sqrt{\frac{n}{n^2+3n^2}} = \sqrt{\frac{1}{4n}} = \frac{1/2}{n^{1/2}}$$

bigger or same

Since $\sum_{n=1}^{\infty} \frac{1/2}{n^{1/2}}$ diverges, the larger $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2+3}}$ also

$\boxed{\text{diverges}}$.

$\boxed{\text{LCT}}$ Compare with divergent $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

$$\lim_{n\to\infty} \frac{\sqrt{\frac{n+1}{n^2+3}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \sqrt{\frac{n^2+n}{n^2+3}} = \sqrt{1} = 1$$ ← between 0 & ∞

Thus both $\boxed{\text{diverge}}$.