MA 126 — Spring 2017 — Prof. Clontz — Standard Assessment 6

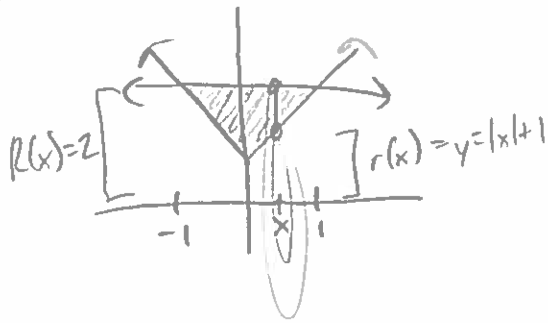
Name: Answers

- Each question is prefaced with a Standard for this course.
- When grading, each response will be marked as follows:
  - ✓: The response is demonstrates complete understanding of the Standard.
  - $\star$: The response may indicate full understanding of the Standard, but clarification or minor corrections are required.
  - $\times$: The response does not demonstrate complete understanding of the Standard.
- Only responses marked with a ✓ mark count toward your grade for the semester. Visit the course website for more information on how to improve $\star$ and $\times$ marks.
- This Assessment is due after 50 minutes. All blank responses will be marked with $\times$.

C07: This student is able to... Use the washer or cylindrical shell method to express a volume of revolution as a definite integral.	Mark: (Instructor Use Only)	Reattempt/ Correction: (Instructor Use Only)

Find a definite integral equal to the volume of the solid of revolution obtained by rotating the region bounded by $y = |x| + 1$ and $y = 2$ around the axis $y = 0$. (Do not solve your integral.)

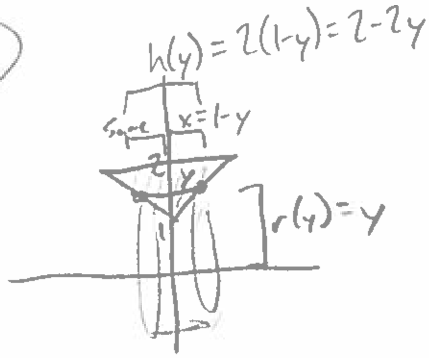
Washer



$$\boxed{\pi \int_{-1}^{1} \left((2)^2 - (|x|+1)^2 \right) dx}$$

OR

Shell



$$\boxed{2\pi \int_{1}^{2} (y)(2-2y)\, dy}$$

C08: This student is able to... Express the work done in a system as a definite integral.	Mark: (Instructor Use Only)	Reattempt/ Correction: (Instructor Use Only)

Hooke's Law states that the force required to compress a spring x units from its natural length requires $F(x) = kx$ units of force for some constant k (depending on the spring). Suppose a spring satisfies $k = 8$ and is naturally length 9. Find a definite integral equal to the work required to compress this spring from length 7 to length 5. (Do not solve your integral.)

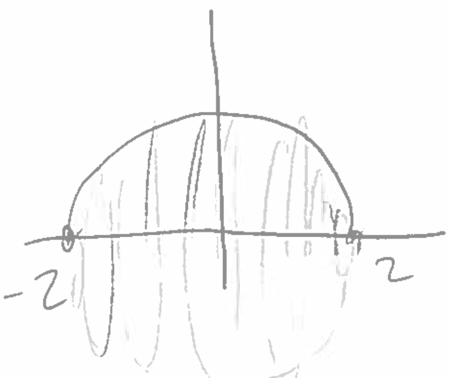
Compressed 2   Compressed 4

$$W = \int_a^b F(x)\,dx$$

$$= \boxed{\int_2^4 8x\,dx}$$

C09: This student is able to... Parametrize a curve to express an arclength or area as a definite integral.	Mark: (Instructor Use Only)	Reattempt/ Correction: (Instructor Use Only)

Recall the following. A smooth curve parametrized by one-to-one functions $x(t), y(t)$ on $a \leq t \leq b$ where $y(t) \geq 0$ may be rotated around the x-axis to yield a surface of revolution. Its area is given by $2\pi \int_a^b y(t)\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt$.
Use this to find a definite integral equal to the surface area of a sphere of radius 2.
(Hint: Rotate a semicircle of radius 2.) (Do not solve your integral.)



$x = 2\cos t \qquad \frac{dx}{dt} = -2\sin t$

$y = 2\sin t \qquad \frac{dy}{dt} = 2\cos t$

$0 \leq t \leq \pi$

$$\boxed{2\pi \int_0^{\pi} 2\sin t \sqrt{(-2\sin t)^2 + (2\cos t)^2}\, dt}$$

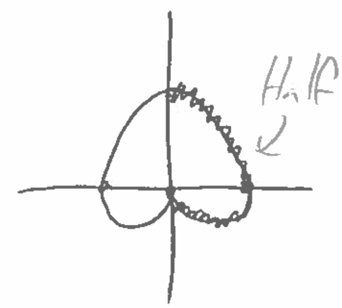
C10: This student is able to... Use polar coordinates to express an arclength or area as a definite integral.	Mark: (Instructor Use Only)	Reattempt/ Correction: (Instructor Use Only)

Find a definite integral equal to half of the circumference of the cardioid $r = 3 + 3\sin(\theta)$ (Do not solve your integral.)

$r' = 3\cos\theta$

$$\text{whole length} = \int_{\alpha}^{\beta} \sqrt{(r(\theta))^2 + (r'(\theta))^2}\, d\theta$$

$$= \int_{0}^{2\pi} \sqrt{(3+3\sin\theta)^2 + (3\cos\theta)^2}\, d\theta$$



$$\text{Half} = \int_{-\pi/2}^{\pi/2} \sqrt{(3+3\sin\theta)^2 + (3\cos\theta)^2}\, d\theta$$

OR

$$\frac{1}{2}\int_{0}^{2\pi} \sqrt{(3+3\sin\theta)^2 + (3\cos\theta)^2}\, d\theta$$

	Mark:	Reattempt/ Correction:
C11: This student is able to... Compute the limit of a convergent sequence.	(Instructor Use Only)	(Instructor Use Only)

Find $\lim_{k\to\infty} \frac{\ln(k^2)+e}{\ln(k)}$. $= \lim_{x\to\infty} \frac{\ln(x^2)+e}{\ln(x)}$

(Using LH)

$= \lim_{x\to\infty} \frac{2x/x^2 + 0}{1/x} = \lim_{x\to\infty} \frac{2x^{\cancel{2}}}{\cancel{x^2}} = \boxed{2}$

(Using algebra)

$= \lim_{k\to\infty} \frac{2\ln(k)+e}{\ln(k)} = \lim_{k\to\infty}\left(\frac{2\cancel{\ln(k)}}{\cancel{\ln(k)}} + \frac{e}{\ln(k)}\right)$

$= 2 + 0 = \boxed{2}$

	Mark:	Reattempt/ Correction:
C12: This student is able to... Express as a limit and find the value of a convergent geometric or telescoping series.	(Instructor Use Only)	(Instructor Use Only)

Find the value of the convergent series $\sum_{k=2}^{\infty}(\frac{1}{2k} - \frac{1}{2k+2})$.

$$= \lim_{n\to\infty}\left(\left(\frac{1}{4} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{8}}\right) + \cdots + \left(\cancel{\frac{1}{2n}} - \frac{1}{2n+2}\right)\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{4} - \frac{1}{2n+2}\right)$$

$$= \frac{1}{4} - 0 = \boxed{\frac{1}{4}}$$

S09: This student is able to... Use parametric equations to find and use tangent slopes.	Mark: (Instructor Use Only)	Reattempt/ Correction: (Instructor Use Only)

Find the point on the parametric curve defined by $x = t^2 + 3$, $y = 4t$ for all real numbers t that has a tangent slope of 1.

$\frac{dx}{dt} = 2t \quad \frac{dy}{dt} = 4$

$\frac{dy}{dx} = 1$

$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

$4 = (1)(2t)$

$t = 2$

$x = 2^2 + 3$
$= 7$

$y = 4(2)$
$= 8$

$\boxed{(7,8)}$

S10: This student is able to... Convert and sketch polar and Cartesian coordinates and equations.	Mark: (Instructor Use Only)	Reattempt/ Correction: (Instructor Use Only)

a) Find a Cartesian coordinate equal to the polar coordinate $p(2, -\pi/4)$.

$r \quad \theta$

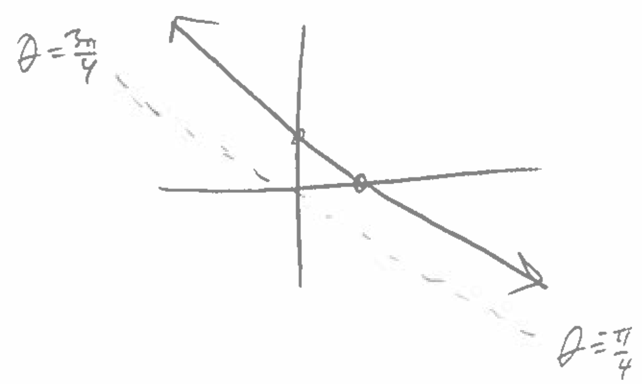
$$x = r\cos\theta$$
$$= 2\cos(-\pi/4)$$
$$= 2\frac{\sqrt{2}}{2}$$
$$= \sqrt{2}$$

$$y = r\sin\theta$$
$$= 2\sin(-\pi/4)$$
$$= 2(-\sqrt{2}/2)$$
$$= -\sqrt{2}$$

$$\boxed{(\sqrt{2}, -\sqrt{2})}$$

b) Sketch the polar curve $r = \frac{1}{\cos\theta + \sin\theta}$ for $-\frac{\pi}{4} < \theta < \frac{3\pi}{4}$ in the xy plane.

$$r\cos\theta + r\sin\theta = 1$$
$$x + y = 1$$
$$y = -x + 1$$



	Mark:	Reattempt/ Correction:
S11: This student is able to... Define and use explicit and recursive formulas for sequences.	(Instructor Use Only)	(Instructor Use Only)

Give an explicit or recursive formula matching the sequence $\langle b_n \rangle_{n=0}^{\infty} = \langle 1, 2, 5, 10, 17, 26, 37, \ldots \rangle$.

Explicit

$= \langle 0+1, 1+1, 4+1, 9+1, 16+1, \cdots \rangle$

$= \boxed{\langle n^2+1 \rangle_{n=0}^{\infty}}$

Recursive

$= \langle 1, 1+1, 2+3, 5+5, 10+7, 17+9, 26+11, \cdots \rangle$

$b_0 = 1$ Base case $\qquad b_{n+1} = b_n + (2n+1)$ prev. + odd

S12: This student is able to... Use the alternating series test to determine series convergence.	Mark: (Instructor Use Only)	Reattempt/ Correction: (Instructor Use Only)

Does $\sum_{m=1}^{\infty}(-1)^m\frac{m+1}{m^2}$ converge or diverge?

Alt. series positive & non-increasing a_m

(Converges if & only if $\lim_{m\to\infty} a_m = 0$.)

$$\lim_{m\to\infty}\frac{m+1}{m^2} = \lim_{m\to\infty}\frac{\cancel{m}}{\cancel{m}}\frac{1+\frac{1}{m}}{m} = 0$$

Thus the series $\boxed{\text{converges}}$.

	Mark:	Reattempt/ Correction:
S13: This student is able to... Use the integral test to determine series convergence.	(Instructor Use Only)	(Instructor Use Only)

a) Does $\int_0^\infty \frac{2e^x}{1+e^{2x}}\,dx$ converge or diverge?
(Hint: $\frac{d}{dx}[\tan^{\leftarrow}(x)] = \frac{1}{1+x^2}$ and $\lim_{x\to\infty} \tan^{\leftarrow}(x) = \pi/2$.)

$$= \lim_{b\to\infty} \int_0^b \frac{2e^x}{1+e^{2x}}\,dx$$

Let $u = e^x$ $\quad x = b \to u = e^b$

$du = e^x dx$ $\quad x = 0 \to u = e^0 = 1$

$$= \lim_{b\to\infty} \int_1^{e^b} \frac{2}{1+u^2}\,du$$

$$= \lim_{b\to\infty} \left[2\tan^{\leftarrow}(e^b) - 2\tan^{\leftarrow} 1\right]$$

$$= 2\left(\pi/2\right) - 2\left(\pi/4\right)$$

$$= \pi - \pi/2 = \pi/2$$

$\boxed{\text{Converges}}$

b) Based on (a), does $\sum_{n=0}^\infty \frac{2e^n}{1+e^{2n}}$ converge or diverge?

$\boxed{\text{Converges}}$ because $\int_0^\infty \frac{2e^x}{1+e^{2x}}\,dx$ does.

	Mark:	Reattempt/ Correction:
S14: This student is able to... Use the ratio and root tests to determine series convergence.	(Instructor Use Only)	(Instructor Use Only)

Does $\sum_{k=3}^{\infty}(\frac{k-3}{k})^{k^2}$ converge or diverge?
(Hint: $\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$.)

$\boxed{\text{Root Test}}$

$$\lim_{k\to\infty} \sqrt[k]{\left|\left(\frac{k-3}{k}\right)^{k^2}\right|} = \lim_{k\to\infty}\left(\frac{k-3}{k}\right)^k$$

$$= \lim_{k\to\infty}\left(1+\frac{-3}{k}\right)^k$$

$$= e^{-3} = \frac{1}{e^3} < 1$$

Thus the series $\boxed{\text{Converges}}$.