MA 126 — Spring 2017 — Prof. Clontz — Standard Assessment 7

Name: Solutions

- Each question is prefaced with a Standard for this course.
- When grading, each response will be marked as follows:
  - ✓: The response is demonstrates complete understanding of the Standard.
  - $\star$: The response may indicate full understanding of the Standard, but clarification or minor corrections are required.
  - $\times$: The response does not demonstrate complete understanding of the Standard.
- Only responses marked with a ✓ mark count toward your grade for the semester. Visit the course website for more information on how to improve $\star$ and $\times$ marks.
- This Assessment is due after 50 minutes. All blank responses will be marked with $\times$.

C09: This student is able to... Parametrize a curve to express an arclength or area as a definite integral.	Mark: (Instructor Use Only)	Reattempt/ Correction: (Instructor Use Only)

Recall that $\cosh(t) = \frac{1}{2}(e^t + e^{-t})$, $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$, and $\cosh^2(t) - \sinh^2(t) = 1$.
Find the arclength of $x^2 - y^2 = 4$ between $(2, 0)$ and $(e + \frac{1}{e}, e - \frac{1}{e})$. (Hint: multiply the hyperbolic identity by 4 on both sides.)
(Do not solve your integral.)

$4\cosh^2 t - 4\sinh^2 t = 4$

$x^2 - y^2 = 4$

$x = 2\cosh t \qquad y = 2\sinh t$

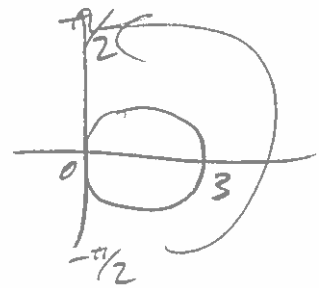
$t=0 \quad x = 2\cosh 0 = 2 \qquad t=1 \quad x = 2\cosh 1 = e^1 + e^{-1}$

$\frac{dx}{dt} = 2\sinh t \qquad \frac{dy}{dt} = 2\cosh t$

$$L = \boxed{\int_0^1 \sqrt{(2\sinh t)^2 + (2\cosh t)^2}\, dt}$$

C10: This student is able to... Use polar coordinates to express an arclength or area as a definite integral.	Mark: (Instructor Use Only)	Reattempt/ Correction: (Instructor Use Only)

Find a definite integral equal to the circumference of the circle $r = 3\cos\theta$. $\frac{dr}{d\theta} = -3\sin\theta$
(Do not solve your integral.)



$$L = \boxed{\int_{-\pi/2}^{\pi/2} \sqrt{(3\cos\theta)^2 + (-3\sin\theta)^2}\, d\theta}$$

Standard Assessment 7

C11: This student is able to... Compute the limit of a convergent sequence.	Mark: (Instructor Use Only)	Reattempt/ Correction: (Instructor Use Only)

Find $\lim_{n\to\infty} \frac{4n+n^4}{5n^4+n^2-3}$. $= \lim_{n\to\infty} \frac{\cancel{n^4}}{\cancel{n^4}} \frac{4/n^3+1}{5+1/n^2-3/n^4} = \boxed{\frac{1}{5}}$

C12: This student is able to... Express as a limit and find the value of a convergent geometric or telescoping series.	Mark: (Instructor Use Only)	Reattempt/ Correction: (Instructor Use Only)

Find the value of the convergent series $\sum_{k=1}^{\infty} 8^{-k}$.

$$= \sum_{k=1}^{\infty} \frac{1}{8^k} \left(= \frac{1}{8} + \frac{1}{64} + \frac{1}{512} + \cdots \right.$$

$$= \sum_{k=0}^{\infty} \frac{1}{8^{k+1}} \left. = \frac{1}{8} + \frac{1}{64} + \frac{1}{512} + \cdots \right)$$

$$= \sum_{k=0}^{\infty} \frac{1}{8}\left(\frac{1}{8}\right)^k \quad \leftarrow \left|\frac{1}{8}\right| < 1$$

$$= \frac{1/8}{1 - 1/8}$$

$$= \frac{1/8}{7/8} = \boxed{\frac{1}{7}}$$

C13: This student is able to... Identify and use appropriate techniques for determining the convergence or divergence of a series.	Mark: (Instructor Use Only)	Reattempt/ Correction: (Instructor Use Only)

Recall the following types of series and techniques for determining series converence.

- Telescoping Series
- Geometric Series
- Alternating Series Test
- Integral Test
- p-Series Test
- Ratio Test
- Root Test
- Comparison Test (Direct/Limit)

Label the following three series with an appropriate type of series or technique for determining series convergence. Then label whether each series converges or diverges (you do not need to show any work).

$$\sum_{k=0}^{\infty} \frac{5}{k^3}$$

p-Series

Converges

$$\sum_{m=1}^{\infty} \frac{2}{(3m)!}$$

Ratio Test

Converges

$$\sum_{n=3}^{\infty} \frac{n}{3^n + 7}$$

Comparison Test

Converges

C14: This student is able to... Identify the domain of a function defined as a power series.	Mark: (Instructor Use Only)	Reattempt/ Correction: (Instructor Use Only)

Prove that $f(x) = \sum_{n=0}^{\infty} \frac{(x-3)^n}{(n+1)!} = 1 + \frac{x-3}{2} + \frac{(x-3)^2}{6} + \frac{(x-3)^3}{24} + \ldots$ is defined for all real numbers x.

Ratio Test

$$\lim_{n\to\infty} \left| \frac{\frac{(x-3)^{n+1}}{(n+2)!}}{\frac{(x-3)^n}{(n+1)!}} \right| = \lim_{n\to\infty} \frac{|x-3|^{n+1}}{(n+2)!} \frac{(n+1)!}{|x-3|^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{|x-3|^n}|x-3|}{\cancel{(n+1)!}(n+2)} \frac{\cancel{(n+1)!}}{\cancel{|x-3|^n}}$$

$$= |x-3| \lim_{n\to\infty} \frac{1}{n+2}$$

$$= |x-3|(0)$$

$$= 0 < 1$$

Thus the series converges for all x.

C15: This student is able to... Generate a Taylor or Maclaurin Series from a function.	Mark: (Instructor Use Only)	Reattempt/ Correction: (Instructor Use Only)

Generate the Maclaurin Series for $\sin(x)$.

$$f^{(0)}(x) = \sin(x) \qquad f^{(0)}(0) = 0$$
$$f^{(1)}(x) = \cos(x) \qquad f^{(1)}(0) = 1$$
$$f^{(2)}(x) = -\sin(x) \qquad f^{(2)}(0) = 0$$
$$f^{(3)}(x) = -\cos(x) \qquad f^{(3)}(0) = -1$$

$$f^{(2k)}(0) = 0$$
$$f^{(2k+1)}(0) = (-1)^k$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \left(\frac{f^{(2k)}(0)}{(2k)!} x^{2k} + \cancel{\frac{f^{(2k+1)}(0)}{(2k+1)!} x^{2k+1}} \right)$$

$$= \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}}$$

S12: This student is able to... Use the alternating series test to determine series convergence.	Mark: (Instructor Use Only)	Reattempt/ Correction: (Instructor Use Only)

Does $\sum_{m=0}^{\infty}(-1)^{m+1}\frac{4+e^m}{e^{m+1}}$ converge or diverge?

positive &
decreasing

$$\lim_{m\to\infty}\frac{4+e^m}{e^{m+1}} = \lim_{m\to\infty}\frac{e^m}{e^m}\,\frac{4e^{-m}+1}{e} = \frac{1}{e} \neq 0$$

Thus the series $\boxed{\text{diverges}}$.

	Mark:	Reattempt/ Correction:
S13: This student is able to... Use the integral test to determine series convergence.	(Instructor Use Only)	(Instructor Use Only)

a) Does $\int_0^\infty \frac{6x^2+4}{x^3+2x+5}\,dx$ converge or diverge?

$= \lim_{b\to\infty} \int_0^b \frac{6x^2+4}{x^3+2x+5}\,dx$

Let $u = x^3+2x+5$ $\quad x=0 \to u=5$

$du = (3x^2+2)dx$ $\quad x=b \to u=b^3+2b+5$

$2du = (6x^2+4)dx$

$= \lim_{b\to\infty} \int_5^{b^3+2b+5} \frac{1}{u}\,du$

$= \lim_{b\to\infty} \ln|b^3+2b+5| - \ln 5$

$\boxed{\text{diverges}}$

b) Based on (a), does $\sum_{n=0}^\infty \frac{6n^2+4}{n^3+2n+5}$ converge or diverge?

It also $\boxed{\text{diverges}}$.

S14: This student is able to... Use the ratio and root tests to determine series convergence.	Mark: (Instructor Use Only)	Reattempt/ Correction: (Instructor Use Only)

Does $\sum_{n=0}^{\infty} \frac{n!n!}{(2n)!}$ converge or diverge?

Ratio Test

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)!\,(n+1)!}{(2n+2)!}}{\frac{n!\,n!}{(2n)!}} \right| = \lim_{n\to\infty} \frac{(n+1)!\,(n+1)!}{(2n+2)!} \frac{(2n)!}{n!\,n!}$$

$$= \lim_{n\to\infty} \frac{\cancel{n!}\,\cancel{n!}\,(n+1)(n+1)}{\cancel{(2n)!}\,(2n+1)(2n+2)} \frac{\cancel{(2n)!}}{\cancel{n!}\,\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2}$$

$$= \frac{1}{4} < 1$$

$\boxed{\text{Converges}}$

S15: This student is able to... Use the comparison tests to determine series convergence.	Mark: (Instructor Use Only)	Reattempt/ Correction: (Instructor Use Only)

Does $\sum_{n=0}^{\infty} \frac{2^n}{3^n+4}$ converge or diverge?

Direct Comparison

$$\frac{2^n}{3^n+4} \leq \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$$

Since $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$ converges, the smaller $\sum_{n=0}^{\infty} \frac{2^n}{3^n+4}$

also $\boxed{\text{converges}}$.

C13b: This student is able to... Identify and use appropriate techniques for determining the convergence or divergence of a series.	Mark: (Instructor Use Only)	Reattempt/ Correction: (Instructor Use Only)

Recall the following types of series and techniques for determining series converence.

- Telescoping Series
- Geometric Series
- Alternating Series Test
- Integral Test
- p-Series Test
- Ratio Test
- Root Test
- Comparison Test (Direct/Limit)

Label the following three series with an appropriate type of series or technique for determining series convergence. Then label whether each series converges or diverges (you do not need to show any work).

$$\sum_{k=0}^{\infty}\left(\frac{3}{k}-\frac{3}{k+1}\right) \qquad \sum_{m=1}^{\infty}\frac{\sin^2(m^3)+1}{m^{2/3}} \qquad \sum_{n=0}^{\infty}\frac{\sqrt{n+1}}{5^n}$$

Telescoping | Comparison | Root

Converges | diverges | converges

C14b: This student is able to... Identify the domain of a function defined as a power series.	Mark: (Instructor Use Only)	Reattempt/ Correction: (Instructor Use Only)

Find the domain of $f(x) = \sum_{n=0}^{\infty}(-\frac{1}{2})^n \frac{x^n}{n^2+1} = 1 - \frac{x}{4} + \frac{x^2}{20} - \frac{x^3}{80} + \ldots$. (You do not need to show your work when determining the convergence/divergence of its endpoints.)

Root Test

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(-\frac{1}{2}\right)^n \frac{x^n}{n^2+1}\right|} = \lim_{n\to\infty} \left(\frac{1}{2}\right) \frac{|x|}{\sqrt[n]{n^2+1}}$$

$$= \frac{1}{2}|x| < 1$$

$$|x| < 2$$

$$-2 < x < 2$$

$\boxed{x = +2}$

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n \frac{2^n}{n^2+1}$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{1}{n^2+1}$$

Converges

$\boxed{x = -2}$

$$\sum_{n=0}^{\infty}\cancel{\left(-\frac{1}{2}\right)^n} \frac{\cancel{(-2)^n}}{n^2+1}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n^2+1}$$

Converges

$$\boxed{-2 \le x \le 2}$$

C15b: This student is able to... Generate a Taylor or Maclaurin Series from a function.	Mark: (Instructor Use Only)	Reattempt/ Correction: (Instructor Use Only)

Generate the Maclaurin Series for 2^x.

$$f^{(0)}(x) = 2^x$$
$$f^{(1)}(x) = 2^x \ln 2$$
$$f^{(2)}(x) = 2^x (\ln 2)^2$$
$$\downarrow$$
$$f^{(k)}(x) = 2^x (\ln 2)^k$$
$$f^{(k)}(0) = (\ln 2)^k$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \boxed{\sum_{k=0}^{\infty} \frac{(\ln 2)^k}{k!} x^k}$$