① Find $\int (x^2-1)(x^2+1)\,dx$.

(Simplify with algebra first.)

$$= \int \left[x^4 \cancel{-2x^2+2x^2} - 1\right] dx$$

$$= \boxed{\frac{1}{5}x^5 - x + C}$$

② Find $\displaystyle\int \frac{1}{\sqrt{9+z^2}}\,dz$.

(No u-sub possible; has $a+bx^2 = a + a\tan^2\theta$ form.)

So use $\boxed{\text{trigonometric substitution}}$.

Let $9+z^2 = 9+9\tan^2\theta = 9\sec^2\theta \longrightarrow \sec^2\theta = 1+\frac{1}{9}z^2$

$z^2 = 9\tan^2\theta$

$z = 3\tan\theta \longrightarrow \tan\theta = z/3$

$dz = 3\sec^2\theta\,d\theta$

$$= \int \frac{1}{\cancel{\sqrt{9\sec^2\theta}}}\,\cancel{3\sec^2\theta}\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \boxed{\ln\left|\sqrt{1+\tfrac{1}{9}z^2} + z/3\right| + C}$$

3. Find $\int 6y^2 e^{y^3}\,dy$.

(y^3 nested in $e^{(\_)}$, with $y^2 dy$: $\boxed{\text{Use u-sub}}$)

Let $u = y^3$

$du = 3y^2\,dy$

$2\,du = 6y^2\,dy$

$= \int 2e^u\,du$

$= 2e^u + C$

$= \boxed{2e^{y^3} + C}$

(4) Find $\int 3x \sin(4x)\,dx$.

(All techniques fail except $\boxed{\text{Int by Parts}}$.)

Let $u = 3x$ $\quad v = -\frac{1}{4}\cos(4x)$

$du = 3\,dx$ $\quad dv = \sin(4x)\,dx$

(To integrate, can use $u = 4x$ if needed)

$$= -\frac{3}{4}x\cos(4x) - \int -\frac{3}{4}\cos(4x)\,dx$$

$$= \boxed{-\frac{3}{4}x\cos(4x) + \frac{3}{16}\sin(4x) + C}$$

(5) Find $\int \sec^3\theta \tan^3\theta \, d\theta$.

(Use $\boxed{\text{trig identities}}$ to substitute $u = \sec\theta$ or $\tan\theta$.)

$$\cancel{= \int \sec\theta \tan^3\theta \left(\sec^2\theta \, d\theta\right)}$$

$\cancel{\text{Let } u = \tan\theta, \; du = \sec^2\theta \, d\theta}$

(← Fails because $\sec\theta$ lacks even power.)

$$= \int \sec^2\theta \tan^2\theta \left(\sec\theta \tan\theta \, d\theta\right)$$

(← Works because $\tan\theta$ has odd power.)

Let $u = \sec\theta$

$du = \sec\theta \tan\theta \, d\theta$

$$= \int \sec^2\theta \left(\sec^2\theta - 1\right)\left(\sec\theta \tan\theta \, d\theta\right)$$

$$= \int u^2 \left(u^2 - 1\right) du$$

$$= \int u^4 - u^2 \, du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$$

$$= \boxed{\frac{1}{5}\sec^5\theta - \frac{1}{3}\sec^3\theta + C}$$

⑥ Find $\int \frac{5x-5}{x^2-3x-4}\,dx$.

$\cancel{\text{Let } u = x^2-3x-4}$
$\cancel{du = (2x-3)\,dx}$
$\cancel{\tfrac{5}{2}\,du = (5x - \tfrac{15}{2})\,dx}$ ← Fails because it doesn't match numerator.

(Try $\boxed{\text{partial fractions}}$..)

$$\frac{5x-5}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1}$$

$$5x-5 = A(x+1) + B(x-4)$$

Let $x=4$

$$20-5 = A(4+1) + 0$$
$$15 = 5A$$
$$A=3$$

Let $x=-1$

$$-5-5 = 0 + B(-1-4)$$
$$-10 = -5B$$
$$B=2$$

$$= \int \frac{3}{x-4} + \frac{2}{x+1}\,dx = \boxed{3\ln|x-4| + 2\ln|x+1| + C}$$

7 Find $\int \left(4\sqrt{t} - 3\tan(t)\sec(t)\right)dt$.

($t^{1/2}$ Power Rule) (derivative of $\sec(t)$)

(Use $\boxed{\text{Calculus 1 techinques}}$:)

$$= 4\left(\frac{2}{3}t^{3/2}\right) - 3\left(\sec(t)\right) + C$$

$$= \boxed{\frac{8}{3}t^{3/2} - 3\sec(t) + C}$$

8 Find $\int e^x \sqrt{1-e^{2x}}\, dx$.

Clever Way

Let $1-e^{2x} = 1-\sin^2\theta = \cos^2\theta \rightarrow \cos\theta = \sqrt{1-e^{2x}}$

$e^{2x} = \sin^2\theta$

$e^x = \sin\theta \longrightarrow \theta = \sin^{\leftarrow}(e^x)$

$e^x dx = \cos\theta\, d\theta$

$= \int \cos\theta \sqrt{\cancel{\cos^2\theta}}\, d\theta$

$= \int \cos^2\theta\, d\theta$

$= \int \frac{1}{2} + \frac{1}{2}\cos 2\theta\, d\theta$

$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$ $\quad \sin 2\theta = 2\sin\theta\cos\theta$

$= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$

$= \boxed{\frac{1}{2}\sin^{\leftarrow}(e^x) + \frac{1}{2}e^x\sqrt{1-e^{2x}} + C}$

$\boxed{\text{OR Longer Way}}$

(First use $\boxed{\text{u-substitution}}$)

Let $u = e^x$

$du = e^x dx$

(Combination of techniques)

$$= \int \sqrt{1-u^2}\, du$$

(Then use $\boxed{\text{trig substitution}}$:)

Let $1-u^2 = 1-\sin^2\theta = \cos^2\theta \rightarrow \cos\theta = \sqrt{1-u^2}$

$u = \sin\theta \rightarrow \theta = \sin^{-1}(u)$

$du = \cos\theta\, d\theta$

$$= \int \sqrt{\cos^2\theta}\, \cos\theta\, d\theta$$

$$= \int \cos^2\theta\, d\theta$$

$$= \ldots = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$$

$$= \frac{1}{2}\sin^{-1}(u) + \frac{1}{2}u\sqrt{1-u^2} + C$$

$$= \boxed{\frac{1}{2}\sin^{-1}(e^x) + \frac{1}{2}e^x\sqrt{1-e^{2x}} + C}$$

(See other page)

9&10 Choose the most appropriate technique to find...

9.1 $\int \frac{4x}{x^2+3}\,dx$

$\boxed{\text{Substitution}}$ (Let $u=x^2+3$, $du=2x\,dx$.)

9.2 $\int \cos^3(x)\,dx$

$\boxed{\text{Trig Identities}}$ $\left(=\int \cos^2(x)\cos(x)\,dx=\int (1-\sin^2(x))\cos(x)\,dx\right)$

9.3 $\int \frac{5}{2x^2+8}\,dx$

$\boxed{\text{Trig Sub}}$ (Let $2x^2+8=8\tan^2\theta+8=8\sec^2\theta$.)

9.4 $\int \frac{x}{\csc(x)}\,dx$

$\boxed{\text{Int by Parts}}$ $\left(=\int x\sin(x)\,dx.\quad \text{Let } \begin{matrix} u=x & v=-\cos x \\ du=dx & dv=\sin(x)\,dx \end{matrix}\right)$

9.5 $\int \frac{4x^2+x+3}{x^3+3x^2}\,dx.$

Partial Fractions $\left(\frac{4x^2+x+3}{(x)^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3} \right)$

10.1 $\int \sec^5(y)\tan^3(y)\,dy.$

Trig Identities

$$\left(\begin{aligned} &= \int \sec^4(y)\tan^2(y)\sec(y)\tan(y)\,dy \\ &= \int \sec^4(y)(1-\sec^2(y))\sec(y)\tan(y)\,dy \end{aligned} \right)$$

10.2 $\int \frac{\sin y}{1-2\cos y}\,dy.$

Substitution $\left(\begin{aligned} \text{Let } u &= 1-2\cos y \\ du &= -2\sin y\,dy \end{aligned} \right)$

10.3 $\int \frac{y^2+4y}{(y^2+4)(y+2)}\,dy.$

Partial Fractions $\left(\frac{y^2+4y}{(y^2+4)(y+2)} = \frac{Ay+B}{y^2+4} + \frac{C}{y+2} \right)$

(10.4) $\int \sqrt{4y^2-9}\,dy$ where $y > 3/2$.

$\boxed{\text{Trig Sub}}$ (Let $4y^2-9 = 9\sec^2\theta - 9 = 9\tan^2\theta$)

(10.5) $\int \cos(y)\sinh(y)\,dy$

$\boxed{\text{Int by Parts}}$ (Let $u = \cos(y)$ $\quad v = \cosh(y)$
$du = -\sin(y)$ $\quad dv = \sinh(y)$) (will req. cycling.)