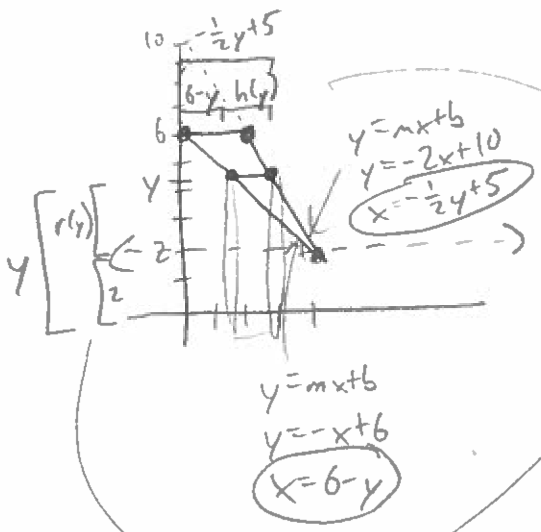
(4) Find the volume of the solid of revolution obtained by rotating the triangle with vertices $(4,2)$, $(2,6)$ $(0,6)$ around the axis $y=2$. (Cylindrical Shell Method)



$r(y)+2=y$

$r(y)=y-2$

$(6-y)+h(y)=-\frac{1}{2}y+5$

$h(y)=\frac{1}{2}y-1$

$$V=2\pi\int_2^6 (y-2)\left(\tfrac{1}{2}y-1\right)dy$$

$$=2\pi\int_2^6 \left(\tfrac{1}{2}y^2-2y+2\right)dy$$

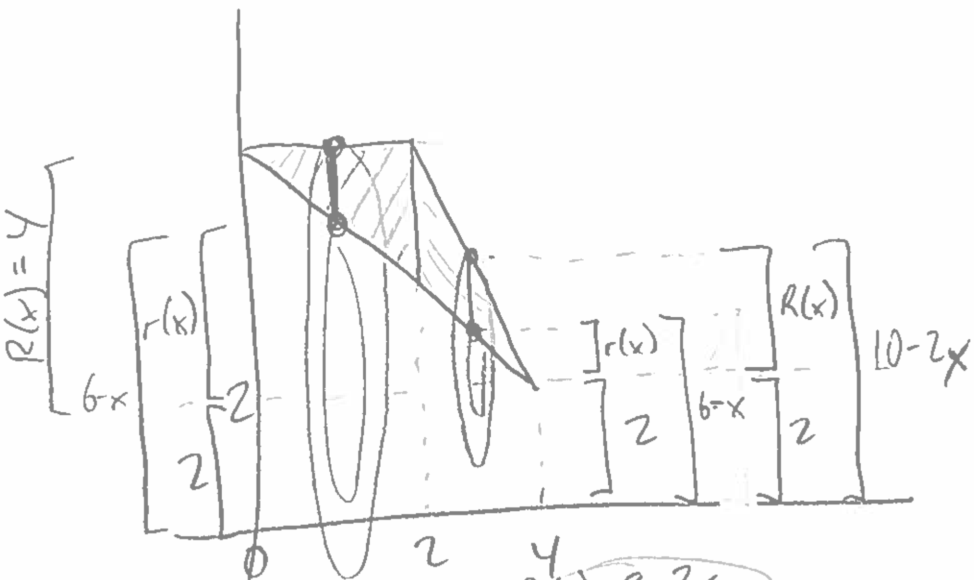
$$=2\pi\left[\tfrac{1}{6}y^3-y^2+2y\right]_2^6$$

$$=2\pi\left[\left(\cancel{36-36}+12\right)-\left(8/6\cancel{-4+4}\right)\right]$$

$$=2\pi\left[\frac{64}{6}\right]=\boxed{\frac{64\pi}{3}}$$

④ using Washer Method

(Split into two pieces...)



$R(x) = 4$
$r(x) = 4 - x$

$R(x) = 8 - 2x$
$r(x) = 4 - x$

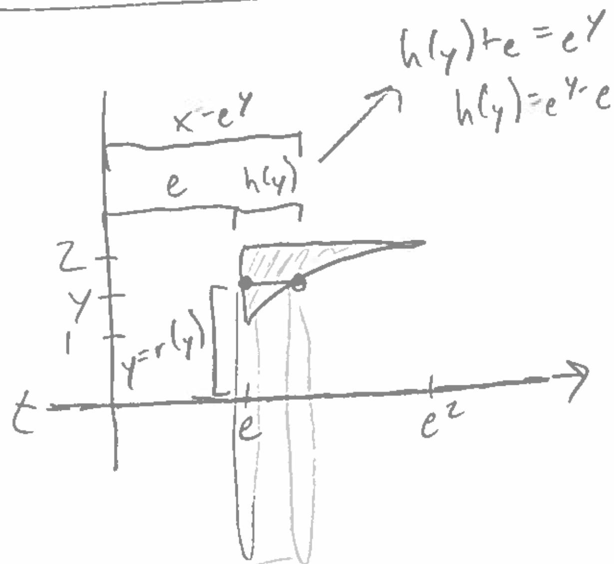
$$V = \pi \int_0^2 \left((4)^2 - (4-x)^2 \right) dx + \pi \int_2^4 \left((8-2x)^2 - (4-x)^2 \right) dx$$

$$= \qquad \vdots \qquad\qquad \vdots$$

$$= \frac{40}{3}\pi + 8\pi$$

$$= \boxed{\frac{64\pi}{3}}$$

(5) Find the volume of the solid of revolution obtained by rotating the region bounded by $x=e$, $y=2$, $y=\ln x$ around the x-axis.



$$V = 2\pi \int_1^2 (y)(e^y - e)\,dy$$

$$= 2\pi \int_1^2 ye^y\,dy - 2\pi e \int_1^2 y\,dy$$

Let $u=y$ $\quad v=e^y$

$du=dy$ $\quad dv=e^y dy$

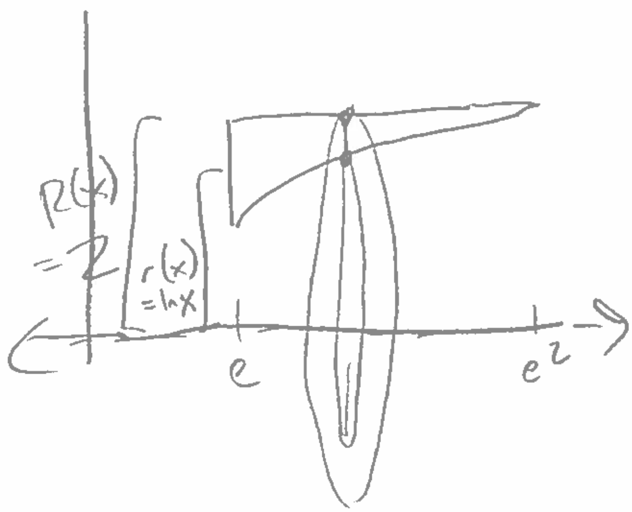
$$= 2\pi\left[ye^y - \int e^y dy\right]_1^2 - 2\pi e\left[\tfrac{1}{2}y^2\right]_1^2$$

$$= 2\pi\left[ye^y - e^y\right]_1^2 - 2\pi e\left[(2)-\left(\tfrac{1}{2}\right)\right]$$

$$= 2\pi\left[\underbrace{(\cancel{2e^2 - e^2})}_{e^2} - \cancel{(e-e)}\right] - 2\pi e\left[\tfrac{3}{2}\right]$$

$$= \boxed{2\pi e^2 - 3\pi e}$$

(5) using Washer Method



$$V = \pi \int_e^{e^2} \left((2)^2 - (\ln x)^2\right) dx$$

$$= 4\pi \int_e^{e^2} dx - \pi \int_e^{e^2} (\ln x)^2 dx$$

$$= \quad \vdots \qquad \vdots$$

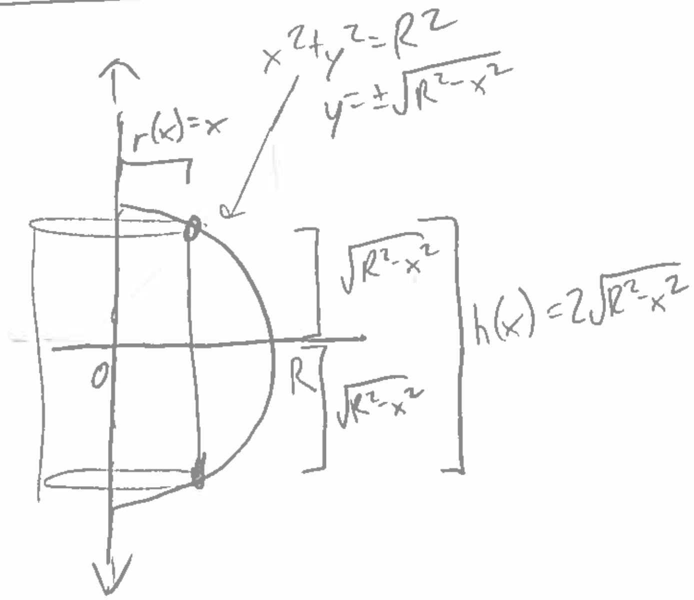
Tricky integration by parts

$$= 4\pi(e^2 - e) - \pi(2e^2 - e)$$

$$= \boxed{2\pi e^2 - 3\pi e}$$

$$\int (\ln x)^2 dx$$
$$= (\ln x)^2 x - \int \cancel{x} \frac{2\ln x}{\cancel{x}} dx$$
$$= (\ln x)^2 x - 2x\ln x + \int 2\cancel{x}\frac{1}{\cancel{x}} dx$$
$$= (\ln x)^2 x - 2x\ln x + 2x + C$$

⑥ Use cylindrical shells to prove $V = \frac{4}{3}\pi R^3$ for a sphere with radius R.



$$V = 2\pi \int_0^R (x)\left(2\sqrt{R^2-x^2}\right)dx$$

$$= 2\pi \int_0^R 2x\sqrt{R^2-x^2}\,dx$$

Let $u = R^2 - x^2$ $\quad$ $x=R \to u=0$

$du = -2x\,dx$ $\quad$ $x=0 \to u=R^2$

$-du = 2x\,dx$

$$= 2\pi \int_{R^2}^{0} -\sqrt{u}\,du$$

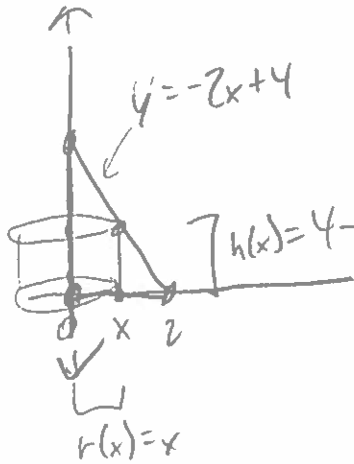
$$= 2\pi \int_0^{R^2} u^{1/2}\,du$$

$$= \frac{4}{3}\pi \left[u^{3/2}\right]_0^{R^2}$$

$$= \frac{4}{3}\pi R^3 - \cancel{\frac{4}{3}\pi 0^3}$$

$$= \frac{4}{3}\pi R^3 \quad \square$$

7. What integral is produced by the cylindrical shell method for the volume of the solid of revolution obtained by rotating the triangle with vertices (0,0), (2,0), (0,4) around the y-axis?



$$\boxed{V = 2\pi \int_0^2 (x)(4-2x)\,dx}$$

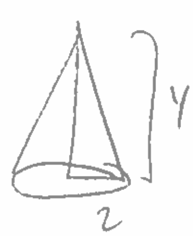
$$= 2\pi \int_0^2 (4x - 2x^2)\,dx$$

$$= 2\pi \left(2x^2 - \tfrac{2}{3}x^3\right]_0^2$$

$$= 2\pi \left[\left(8 - \tfrac{16}{3}\right) - \cancel{(0-0)}\right]$$

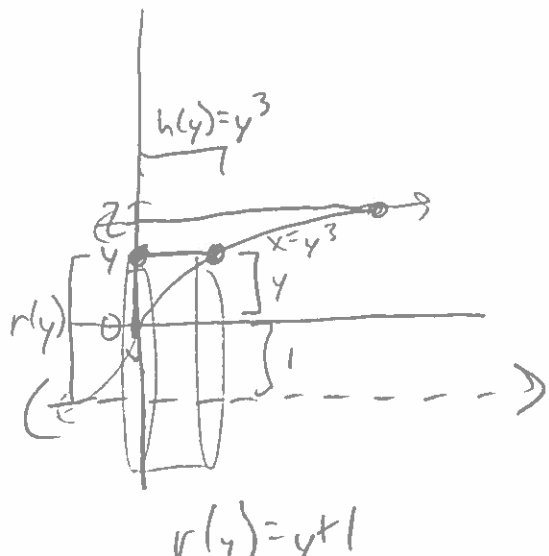
$$= \frac{16\pi}{3}$$

Also, this is a cone of height 4 & radius 2:



$$V = \tfrac{1}{3}\pi r^2 h$$
$$= \tfrac{1}{3}\pi (2)^2 (4)$$
$$= \frac{16\pi}{3}$$

(8) What integral is produced by cylindrical shells for the volume of the solid of revolution obtained by rotating the region bounded by $x=0$, $y=2$, $x=y^3$ around the axis $y=-1$?



$$\boxed{V = 2\pi \int_0^2 (y+1)(y^3)\,dy}$$

$$= 2\pi \int_0^2 y^4 + y^3 \,dy$$

$$= 2\pi \left[\frac{1}{5}y^5 + \frac{1}{4}y^4\right]_0^2$$

$$= 2\pi \left[\left(\frac{32}{5} + 4\right) - (0+0)\right]$$

$$= \frac{104\pi}{5}$$