1. Expand $\sum_{n=0}^{\infty} 3^{n+1} x^n$.

$$= 3^{0+1}x^0 + 3^{1+1}x^1 + 3^{2+1}x^2 + 3^{3+1}x^3 + \cdots$$

$$= \boxed{3 + 9x + 27x^2 + 81x^3 + \cdots}$$

2. Expand $\sum_{k=1}^{\infty} \frac{(-x)^k}{k+1}$.

$$= \frac{(-x)^1}{1+1} + \frac{(-x)^2}{2+1} + \frac{(-x)^3}{3+1} + \frac{(-x)^4}{4+1} + \cdots$$

$$= \boxed{-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \cdots}$$

3. Expand $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$.

$$= (-1)^0 \frac{x^1}{1!} + (-1)^1 \frac{x^3}{3!} + (-1)^2 \frac{x^5}{5!} + (-1)^3 \frac{x^7}{7!} + \cdots$$

$$= \boxed{x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots}$$

(4) Simplify $q(x)=\sum_{n=1}^{\infty}(1-x)^{n}$ with domain $|1-x|<1$.

$$= \sum_{n=0}^{\infty}(1-x)^{n+1}$$

$$= \sum_{n=0}^{\infty}\underset{a}{\underset{\uparrow}{(1-x)}}\underset{r}{\underset{\uparrow}{(1-x)}}^{n}$$

Since $|r| = |1-x| < 1$,

$$= \frac{1-x}{1-(1-x)} = \frac{1-x}{\cancel{1-1}+x} = \frac{1}{x} - \frac{x}{x} = \boxed{\frac{1}{x}-1}.$$

(5) Simplify $g(x)=\sum_{j=0}^{\infty}(2x)^{2j+1}$ with domain $|x|<\frac{1}{2}$.

$$= \sum_{j=0}^{\infty}(2x)^{1}(2x)^{2j}$$

$$= \sum_{j=0}^{\infty}(2x)(4x^{2})^{j}$$

Since $|r| = |4x^2| = 4|x|^2 < 4\left(\frac{1}{2}\right)^2 = 4\left(\frac{1}{4}\right) = 1$,

$$= \boxed{\frac{2x}{1-4x^{2}}}$$

⑥ Find the domain of $z(x) = \sum_{n=2}^{\infty} (-1)^n \frac{x^n}{2n}$.

**Root Test**

$$\lim_{n\to\infty} \sqrt[n]{\left|\cancel{(-1)^n}\frac{x^n}{2n}\right|} = \lim_{n\to\infty} \frac{|x|^{\cancel{n/n}}}{\sqrt[n]{2n}} = |x| \lim_{n\to\infty}\frac{1}{\sqrt[n]{2n}}$$

$$= |x|(1) = |x| < 1$$

$$-1 < x < 1$$

**Endpoints**

**$x = -1$**

$$= \sum_{n=2}^{\infty} \cancel{(-1)^n} \frac{\cancel{(-1)^n}}{2n}$$

$$= \sum_{n=2}^{\infty} \frac{1/2}{n}$$

divergent p-Series

**$x = 1$**

$$= \sum_{n=2}^{\infty} (-1)^n \frac{\cancel{(1)^n}}{2n}$$

$$= \sum_{n=2}^{\infty} (-1)^n \frac{1}{2n} \quad \& \quad \lim_{n\to\infty} \frac{1}{2n} = 0$$

Alt Series

positive & nonincreasing

Convergent by Alt. Series Test

Domain: $\boxed{-1 < x \leq 1}$

7) Find the domain of $\sum_{i=0}^{\infty} \frac{(3x)^i}{(2i)!}$.

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Ratio Test

$$\lim_{i\to\infty} \left| \frac{\frac{(3x)^{i+1}}{(2(i+1))!}}{\frac{(3x)^i}{(2i)!}} \right| = \lim_{i\to\infty} \frac{|3x|^{i+1}}{(2i+2)!} \cdot \frac{(2i)!}{|3x|^i}$$

$$= \lim_{i\to\infty} \frac{|3x|\cancel{|3x|^i}}{\cancel{(2i)!}(2i+1)(2i+2)} \cdot \frac{\cancel{(2i)!}}{\cancel{|3x|^i}}$$

$$= |3x| \lim_{i\to\infty} \frac{1}{(2i+1)(2i+2)}$$

$$= |3x|(0) = \underbrace{0 < 1}_{\text{true for all } x}$$

Domain: $\boxed{\text{all real \#s}}$

OR

$\boxed{-\infty < x < \infty}$

(8) Find the domain of $h(x) = \sum_{k=0}^{\infty} \frac{(x-2)^k}{k^2+1}$.

Root Test

$$\lim_{k\to\infty} \sqrt[k]{\left|\frac{(x-2)^k}{k^2+1}\right|} = \lim_{k\to\infty} \frac{|x-2|^{\cancel{k}}}{\sqrt[k]{k^2+1}} = |x-2| < 1$$

$$-1 < x - \cancel{2} < 1$$
$$+2 \quad +\cancel{2} \quad +2$$

$$1 < x < 3$$

Endpoints

$x = 1$

$$= \sum_{k=0}^{\infty} \frac{(1-2)^k}{k^2+1}$$
$$= \sum_{k=0}^{\infty} (-1)^k \frac{1}{k^2+1}$$

converges by A.S.T.

$x = 3$

$$= \sum_{k=0}^{\infty} \frac{(3-2)^k}{k^2+1}$$
$$= \sum_{k=0}^{\infty} \frac{\cancel{(1)^k}}{k^2+1} = \sum_{k=0}^{\infty} \frac{1}{k^2+1}$$

converges by DCT, LCT, Integral Test, (take your pick)

Domain: $\boxed{1 \le x \le 3}$

4. Find the domain of $g(x) = \sum_{n=3}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) x^n$.

$$= \sum_{n=3}^{\infty} \left(\frac{1}{n^2+n}\right) x^n$$

**Root Test**

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{x^n}{n^2+n}\right|} = \lim_{n\to\infty} \frac{|x|^{\cancel{n}/\cancel{n}}}{\sqrt[n]{n^2+n}} = |x| < 1$$

$-1 < x < 1$

**Endpoints**

$\boxed{x = -1}$ $= \sum_{n=3}^{\infty} (-1)^n \frac{1}{n^2+n}$

Converges by A.S.T.

$\boxed{x = 1}$ $= \sum_{n=3}^{\infty} \frac{1}{n^2+n}$

$$= \sum_{n=3}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Converges (to $1/3$) as a telescoping series (or use DCT/LCT)

Domain: $\boxed{-1 \le x \le 1}$

(10) Expand $\displaystyle\sum_{k=0}^{\infty} \frac{(-x)^{2k+1}}{(2k)!}$.

$$= \frac{(-x)^1}{0!} + \frac{(-x)^3}{2!} + \frac{(-x)^5}{4!} + \frac{(-x)^7}{6!} + \cdots$$

$$= \boxed{-x - \frac{x^3}{2} - \frac{x^5}{24} - \frac{x^7}{720} - \cdots}$$

(11) Simplify $f(x) = \displaystyle\sum_{n=1}^{\infty} (-x)^{n-1}$ with domain $|x| < 1$.

$$= \sum_{n=0}^{\infty} (-x)^{n\cancel{+1-1}}$$

$$= \sum_{n=0}^{\infty} (1)(-x)^n$$

Since $|r| = |-x| = |x| < 1$,

$$= \frac{1}{1-(-x)} = \boxed{\frac{1}{1+x}}$$

(12) Find the domain of $f(x) = \sum_{n=2}^{\infty} \frac{(-2x)^n}{n}$.

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$\boxed{\text{Root Test}}$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-2x)^n}{n}\right|} = \lim_{n\to\infty} \frac{|2x|^{\cancel{n/n}}}{\sqrt[n]{n}} = \frac{|2x|}{1} < 1$$

$$-1 < 2x < 1$$

$$-\tfrac{1}{2} < x < \tfrac{1}{2}$$

$\boxed{\text{Endpoints}}$

$\boxed{x = -1/2}$ $= \sum_{n=2}^{\infty} \frac{\cancel{(-2(-1/2))^n}}{n}$

$= \sum_{n=2}^{\infty} \frac{1}{n}$

diverges as Harmonic Series

$\boxed{x = 1/2}$ $= \sum_{n=2}^{\infty} \frac{(-2(\frac{1}{2}))^n}{n}$

$= \sum_{n=2}^{\infty} (-1)^n \frac{1}{n}$

converges by A.S.T.

Domain: $\boxed{-\tfrac{1}{2} < x \le \tfrac{1}{2}}$