① Show that the Maclaurin series generated by $f(x) = \frac{1}{1-x}$, $-1 < x < 1$ converges to $f(x)$.

$$f^{(0)}(x) = \frac{1}{1-x} = (1-x)^{-1} \rightarrow f^{(0)}(0) = 1^{-1} = 1$$

$$f^{(1)}(x) = +(1-x)^{-2} \rightarrow f^{(1)}(0) = 1^{-2} = 1$$

$$f^{(2)}(x) = +2(1-x)^{-3} \rightarrow f^{(2)}(0) = 2(1)^{-3} = 2$$

$$f^{(3)}(x) = +6(1-x)^{-4} \rightarrow f^{(3)}(0) = 6(1)^{-4} = 6$$

$$f^{(4)}(x) = +24(1-x)^{-5} \rightarrow f^{(4)}(0) = 24(1)^{-5} = 24$$

$$\downarrow$$

$$f^{(k)}(0) = k!$$

$$\text{Mac Series} = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$$= \sum_{k=0}^{\infty} \frac{\cancel{k!}}{\cancel{k!}} x^k = \boxed{\sum_{k=0}^{\infty} x^k}$$

This series converges to...

$$= \sum_{k=0}^{\infty} \underset{a}{(1)} \underset{|r| = |x| < 1}{(x)^k} = \frac{1}{1-x} = f(x). \quad \square$$

② Show that the Taylor series generated by $g(x) = \frac{3}{x}$, $0<x<6$ at $x=3$ converges to $g(x)$.

$$g^{(0)}(x) = 3x^{-1}$$
$$g^{(1)}(x) = 3(-1)x^{-2}$$
$$g^{(2)}(x) = 3(+2)x^{-3} \longrightarrow$$
$$g^{(3)}(x) = 3(-6)x^{-4}$$
$$g^{(4)}(x) = 3(24)x^{-5}$$

$$g^{(0)}(3) = 3(3)^{-1} = +1$$
$$g^{(1)}(3) = 3(-1)(3)^{-2} = -\frac{1}{3}$$
$$g^{(2)}(3) = 3(+2)(3)^{-3} = +\frac{2}{3^2}$$
$$g^{(3)}(3) = 3(-6)(3)^{-4} = -\frac{6}{3^3}$$
$$g^{(4)}(3) = 3(24)(3)^{-5} = +\frac{24}{3^4}$$
$$\downarrow$$
$$g^{(k)}(3) = (-1)^k \frac{k!}{3^k} = \left(-\tfrac{1}{3}\right)^k k!$$

$$\text{Taylor series} = \sum_{k=0}^{\infty} \frac{f^{(k)}(3)}{k!}(x-3)^k = \sum_{k=0}^{\infty} \frac{(-1/3)^k \cancel{k!}}{\cancel{k!}}(x-3)^k$$
$$= \boxed{\sum_{k=0}^{\infty} (-1/3)^k (x-3)^k}$$

This series converges to...

$$= \sum_{k=0}^{\infty}\left(-\frac{x}{3}+1\right)^k = \frac{1}{1-\left(-\frac{x}{3}+1\right)} = \frac{1}{x/3} = \frac{3}{x} \qquad \square$$

$$\uparrow$$
$$\left|-\frac{x}{3}+1\right| < 1$$
$$|x-3| < 3$$
$$-3 < x-3 < 3$$
$$0 < x < 6$$

③ Show that the Maclaurin series for $h(x) = \frac{1}{1+x^2}$, $-1<x<1$ converges to $h(x)$, given $\left\langle h^{(k)}(0)\right\rangle_{k=0}^{\infty} = \langle 1, 0, -2, 0, 24, 0, -720, \ldots\rangle$

$$h^{(2k+1)}(0) = 0 \qquad h^{(2k)}(0) = (-1)^k (2k)!$$

$$\text{Mac Series} = \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} x^k$$

$$= \sum_{k=0}^{\infty} \frac{h^{(2k)}(0)}{(2k)!} x^{2k} + \cancel{\sum_{k=0}^{\infty} \frac{h^{(2k+1)}(0)}{(2k+1)!} x^{2k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \cancel{(2k)!}}{\cancel{(2k)!}} x^{2k} = \boxed{\sum_{k=0}^{\infty} (-1)^k x^{2k}}$$

This series converges to...

$$= \sum_{k=0}^{\infty} (-1)^k (x^2)^k = \sum_{k=0}^{\infty} (-x^2)^k = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2} \quad \square$$

$$|-x^2| < 1$$
$$x^2 < 1$$
$$-1 < x < 1$$

4 Generate the Mac series for cos x.

$$f^{(0)}(x) = \cos x \qquad f^{(0)}(0) = 1$$
$$f^{(1)}(x) = -\sin x \qquad f^{(1)}(0) = 0 \quad \rightarrow \quad f^{(2k)}(0) = (-1)^k$$
$$f^{(2)}(x) = -\cos x \qquad f^{(2)}(0) = -1 \qquad f^{(2k+1)}(0) = 0$$
$$f^{(3)}(x) = \sin x \qquad f^{(3)}(0) = 0$$

$$\text{Mac series} = \sum_{k=0}^{\infty} \frac{f^{(2k)}(0)}{(2k)!} x^{2k} + \cancel{\sum_{k=0}^{\infty} \frac{f^{(2k+1)}(0)}{(2k+1)!} x^{2k+1}}$$

$$= \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}}$$

(5) Find the Mac Series for $\sinh x$.

$$f^{(0)}(x) = \sinh x \rightarrow f^{(0)}(0) = 0 \rightarrow f^{(2k)}(0) = 0$$

$$f^{(1)}(x) = \cosh x \qquad f^{(1)}(0) = 1 \qquad f^{(2k+1)}(0) = 1$$

$$\text{Mac Series} = \cancel{\sum_{k=0}^{\infty} \frac{f^{(2k)}(0)}{(2k)!} x^{2k}} + \sum_{k=0}^{\infty} \frac{\cancel{f^{(2k+1)}(0)}}{(2k+1)!} x^{2k+1}$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}}$$

(6) Find the Mac Series for $\cosh x$.

$f^{(0)}(x) = \cosh x \rightarrow f^{(0)}(0) = 1 \rightarrow f^{(2k)}(0) = 1$

$f^{(1)}(x) = \sinh x \rightarrow f^{(1)}(0) = 0 \rightarrow f^{(2k+1)}(0) = 0$

$$\text{Mac Series} = \sum_{k=0}^{\infty} \frac{\cancel{f^{(2k)}(0)}}{(2k)!} x^{2k} + \cancel{\sum_{k=0}^{\infty} \frac{f^{(2k+1)}(0)}{(2k+1)!} x^{2k+1}}$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}}$$

2) Find the Maclaurin series for e^{-x}.

$f^{(0)}(x) = e^{-x}$

$f^{(1)}(x) = -e^{-x} \rightarrow f^{(0)}(0) = 1$

$f^{(1)}(0) = -1 \rightarrow f^{(k)}(0) = (-1)^k$

$$\text{Mac Series} = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$$= \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}}$$

⑧ Find the Mac Series for x^3+3x-7.

$f^{(0)}(x) = x^3+3x-7$ $\qquad f^{(0)}(0) = -7$

$f^{(1)}(x) = 3x^2+3$ $\qquad f^{(1)}(0) = 3$

$f^{(2)}(x) = 6x \rightarrow$ $\qquad f^{(2)}(0) = 0 \rightarrow$ $\qquad f^{(k)}(0) = 0$ for $k \geq 4$

$f^{(3)}(x) = 6$ $\qquad f^{(3)}(0) = 6$

$f^{(4)}(x) = 0$ $\qquad f^{(4)}(0) = 0$

$$\text{Mac Series} = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$$= \frac{-7}{0!}x^0 + \frac{3}{1!}x^1 + \cancel{\frac{0}{2!}x^2} + \frac{\cancel{6}}{\cancel{3!}}x^3 + \cancel{\sum_{k=4}^{\infty} \frac{f^{(k)}(0)}{k!}} x^k$$

$$= \boxed{-7 + 3x \qquad + x^3}$$