① Find a power series converging to

$\frac{x^2}{1-x}$ for $-1<x<1$.

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1+x+x^2+x^3+\dots$$

$$\frac{x^2}{1-x} = x^2 \sum_{k=0}^{\infty} x^k$$

$$= \boxed{\sum_{k=0}^{\infty} x^{k+2} = x^2+x^3+x^4+x^5+\dots}$$

② Find a power series converging to

$x^3 \cos x$ for all x.

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$x^3 \cos x = x^3 \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$= \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+3}}{(2k)!} = x^3 - \frac{x^5}{2!} + \frac{x^7}{4!} - \frac{x^9}{6!} + \dots}$$

③ Find the Taylor series generated by $(x-\pi/2)\sin x$ at $\pi/2$.

$$\begin{aligned} f^{(0)}(x) &= \sin x \\ f^{(1)}(x) &= \cos x \\ f^{(2)}(x) &= -\sin x \\ f^{(3)}(x) &= -\cos x \end{aligned} \rightarrow \begin{aligned} f^{(0)}(\pi/2) &= 1 \\ f^{(1)}(\pi/2) &= 0 \\ f^{(2)}(\pi/2) &= -1 \\ f^{(3)}(\pi/3) &= 0 \end{aligned} \rightarrow \begin{aligned} f^{(2k)}(\pi/2) &= (-1)^k \\ f^{(2k+1)}(\pi/2) &= 0 \end{aligned}$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{f^{(2k)}(\pi/2)}{(2k)!}(x-\pi/2)^k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}(x-\pi/2)^k$$

$$(x-\pi/2)\sin(x) = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}(x-\pi/2)^{k+1}} = (x-\pi/2) - \frac{(x-\pi/2)^3}{2!} + \frac{(x-\pi/2)^5}{4!} - \frac{(x-\pi/2)^7}{6!} + \dots$$

④ Find the Maclaurin series generated by $\sin(x^3)$.

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin(x^3) = \sum_{k=0}^{\infty} (-1)^k \frac{(x^3)^{2k+1}}{(2k+1)!}$$

$$= \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{6k+3}}{(2k+1)!}} = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \cdots$$

⑥ Find a power series converging to $\frac{2}{2-x}$ for $-2 < x < 2$.

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad \text{for } -1 < x < 1$$

$$\frac{2}{2-2x} = \sum_{k=0}^{\infty} x^k$$

$$\frac{2}{2-2(\frac{1}{2}x)} = \sum_{k=0}^{\infty} \left(\tfrac{1}{2}x\right)^k \quad \text{for } -1 < \tfrac{1}{2}x < 1 \Rightarrow -2 < x < 2$$

$$= \boxed{\sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k x^k} = 1 + x/2 + x^2/4 + x^3/8 + x^4/16 + \cdots$$

(8) Find the Maclaurin series generated by $3x^2\cos(x^3)$.

$$\sin(x^3) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{6k+3}}{(2k+1)!} = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \cdots$$

$$3x^2\cos(x^3) = \frac{d}{dx}\left[\sum_{k=0}^{\infty} (-1)^k \frac{x^{6k+3}}{(2k+1)!}\right]$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(6k+3)}{(2k+1)!} x^{6k+2}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{3\cancel{(2k+1)}}{(2k)!\cancel{(2k+1)}} x^{6k+2}$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k 3}{(2k)!} x^{6k+2}} = 3x^2 - \frac{3x^8}{2!} + \frac{3x^{14}}{4!} - \frac{3x^{20}}{6!} + \cdots$$

(7) Find a power series converging to $\tan^{-1}(x)$ for $-1<x<1$.

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1+x+x^2+x^3+\cdots$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{k=0}^{\infty}(-x^2)^k$$

$$= \sum_{k=0}^{\infty}(-1)^k x^{2k} = 1-x^2+x^4-x^6+\cdots$$

$$\tan^{-1}(x) = \int \frac{1}{1+x^2}\,dx$$

$$= \int \sum_{k=0}^{\infty}(-1)^k x^{2k}\,dx$$

$$= \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1} + C$$

$$\tan^{-1}(0) = \cancel{\sum_{k=0}^{\infty}(-1)^k \frac{0^{2k+1}}{2k+1}} + C = 0$$

$$C=0$$

$$\tan^{-1}(x) = \boxed{\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$